11. (12 points) (See problems 10 and 11 from Chapter 4, Section 4.) Ms. Manufacturer, a producer of diamond-studded widgets, finds that her company can sell q widgets per week if they are priced at p each, where

$$q = 100 - 2p$$

and p is in hundreds of dollars. Her cost, also measured in hundreds of dollars, for producing q widgets is

$$C(q) = 100 + 10q + \frac{1}{2}q^2.$$

(a) How many widgets should her company manufacture each week to achieve the least cost per widget? That is, the least average cost. (Be sure to show your work.)

Solution: The average cost is $a(q) = C(q)/q = 100/q + 10 + .5q$. To minimize $a(q)$, we look for points where $a'(q) = 0$,

$$0 = a'(q) = \frac{-100}{q^2} + \frac{1}{2} \quad \textbf{or} \quad q^2 = 200.$$

Therefore, $q = \sqrt{200} \simeq 14.14$ is the only critical point of the average cost function for $q > 0$. Further, since the average cost $a(q)$ is positive and $a(q) \to +\infty$ as $q \to 0$ or $q \to +\infty$, this critical point must be a global minimum of $a(q)$ on $q > 0$. Thus, she should produce about 14 widgets per week to achieve the least average cost.

(b) Determine the formula for the revenue $R(q)$ received each week if q widgets are sold.

$$R(q) = \textbf{price} \cdot \textbf{quantity} \quad = \left(50 - \frac{q}{2}\right) q = 50q - \frac{q^2}{2}$$

(c) How many widgets should Ms. Manufacturer's company produce each week in order to maximize profits? At what price should the widgets be sold?

$$\textbf{Profit} = \pi(q) = \textbf{Revenue} - \textbf{Cost} = (50q - \frac{q^2}{2}) - (100 + 10q + \frac{1}{2}q^2)$$

At the maximum of $\pi(q)$,

$$0 = \pi'(q) = R'(q) - C'(q) = (50 - q) - (10 + q) = 40 - 2q \quad \textbf{or} \quad q = 20.$$

Since $\pi''(q) = -2 < 0$ at $q = 20$, this must be a local maximum, and since it is the only critical point in $q > 0$ it must be a global maximum. The profit will be maximized when 20 widgets are produced each week and sold for a price of $50 - \frac{20}{2} = 40$, or \$4,000 apiece.