2. (7 points) What is the largest area a rectangle can have if its base lies on the $x$-axis and its upper vertices lie on the curve $y = a^2 - x^2$? (Your answer will be in terms of $a$. Show your work.)

Solution: The rectangle has length $l = 2x$ and width $w = a^2 - x^2$, so we want to maximize the area

$$A = lw = 2x(a^2 - x^2) = 2a^2x - 2x^3,$$

for $0 \leq x \leq a$.

Because $A = 2a^2x - x^3$, we have $A' = 2a^2 - 6x^2$, so that if $A' = 0$ then $x^2 = \frac{a^2}{3}$ and $x = \frac{|a|}{\sqrt{3}} = \frac{a}{\sqrt{3}}$ since $x \geq 0$.

Note that this is the only critical point of $A$ for $0 \leq x \leq a$. At the endpoints of this interval, $A(0) = 0$ and $A(a) = 0$. The value of $A$ at the critical point is positive, $A(\frac{|a|}{\sqrt{3}}) = \left( \frac{2a^2}{\sqrt{3}} - 2 \left( \frac{a}{\sqrt{3}} \right)^3 \right) = \frac{2a^2}{\sqrt{3}} \left( a^2 - \frac{2a^2}{3} \right) = \frac{2a^2}{3\sqrt{3}} > 0$. Since there is only one critical point in the interval, this value of $A$ must be the global maximum for $0 < x < a$. 