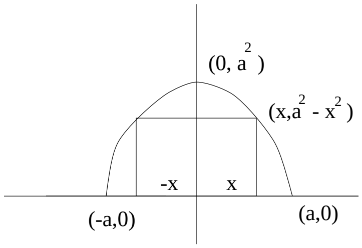
2. (7 points) What is the largest area a rectangle can have if its base lies on the x-axis and its upper vertices lie on the curve $y = a^2 - x^2$? (Your answer will be in terms of a. Show your work.).



Solution: The rectangle has length $l = 2x$ and width $w = a^2 - x^2$, so we want to maximize the area

$$A = lw = 2x(a^2 - x^2) = 2a^2x - 2x^3, \qquad \text{for } \; 0 \leq x \leq a.$$

Because $A = 2a^2x - wx^3$, we have $A' = 2a^2 - 6x^2$, so that if $A' = 0$ then $x^2 = \frac{a^2}{3}$ and $x = \frac{|a|}{\sqrt{3}} = \frac{a}{\sqrt{3}}$ since $x \geq 0$.

Note that this is the only critical point of A for $0 \leq x \leq a$. At the endpoints of this interval, $A(0) = 0$ and $A(a) = 0$. The value of A at the critical point is positive, $A(\frac{|a|}{\sqrt{3}}) = \left((\frac{2a \cdot a^2}{\sqrt{3}} - 2\left(\frac{a}{\sqrt{3}}\right)^3 \right) = \frac{2a}{\sqrt{3}}\left(a^2 - \frac{a^2}{3}\right)\frac{2a}{\sqrt{3}}\left(\frac{2a^2}{3}\right) = \frac{4a^3}{3\sqrt{3}} > 0$. Since there is only one critical point in the interval, this value of A must be the global maximum for $0 < x < a$.