5. (10 points) (See problem 11, Chapter 4, Section 2.) Consider the family of functions defined for $x \geq 0$ by $y = axe^{-bx}$ where a and b are positive numbers. Find all members of this family that pass through the point $(2, 3)$ and have a critical point at $x = 2$. Determine if this critical point is a local maximum, local minimum, or neither. If it is a local maximum or minimum, is it also a global maximum or minimum on this domain (i.e. for $x \geq 0$)?

Solution: Let $y = f(x) = axe^{-bx}$. We are given that $f(2) = 3$ and $f'(2) = 0$. First calculate $f'(x)$.

$$f'(x) = ax\left(-be^{-bx}\right) + ae^{-bx} = ae^{-bx}\left(-bx + 1\right)$$

If $f'(x) = 0$, then we must have $bx = 1$. Since this holds for $x = 2$, $2b = 1$ or $b = \frac{1}{2}$.

Next use $f(2) = 3$ and $b = \frac{1}{2}$ to see that

$$3 = f(2) = 2ae^{-1} = \frac{2a}{e} \quad \textbf{or} \quad a = \frac{3e}{2}.$$

Thus, $y = \frac{3e}{2}xe^{-\frac{1}{2}x} = \frac{3}{2}xe^{(1-\frac{x}{2})}$.

Note: We saw that $x = 2$ is the only critical point of the function. Also, $f'(x) > 0$ for $x < \frac{1}{b}$, (or $x < 2$) and $f'(x) < 0$ for $x > \frac{1}{b}$, (or $x > 2$) from the formula derived for $f'(x)$. Thus, $x = \frac{1}{b}$ (or $x = 2$) gives a local maximum. Since $f(x) \geq 0$, $f(0) = 0$, and $f(x) \to 0$ as $x \to +\infty$, $x = \frac{1}{b}$ (or $x = 2$) is also a global maximum for $x \geq 0$.