8. (8 points) (See Team problem 16, Chapter 4, Section 4, as well as problems 29 and 31 from Chapter 4, Section 3.) (a) It is a fact from economics that when the average cost \( C(q)/q \) of producing \( q > 0 \) units of a quantity is a minimum, then this average cost is equal to the marginal cost. Show analytically why this is so.

Solution: At a point where the average cost \( a(q) = C(q)/q \) is a minimum the derivative of \( a(q) \) must be equal to 0. By the quotient rule,

\[
a'(q) = \frac{q C'(q) - C(q)}{q^2}, \quad q > 0
\]

so \( a'(q) = 0 \) if and only if \( q C'(q) - C(q) = 0 \), or \( C'(q) = \frac{C(q)}{q} \). Thus, at the critical points of \( a(q) \), the marginal cost, \( C'(q) \), is equal to the average cost \( \frac{C(q)}{q} \).

(b) Using the graph of \( C(q) \) shown below, a typical cost function as drawn in the text, indicate on the \( q \)-axis the value of \( q_0 \) which minimizes the average cost. Explain graphically why the average cost is equal to the marginal cost at this point.

The average cost \( a(q) = \frac{C(q)}{q} \) is the slope of the line from the origin to a point \( (q, C(q)) \) on the graph of \( C \). The line with minimal slope is shown above. At this point, \( q_0 \), the line from the origin is tangent to the curve so its slope is also equal to \( C'(q_0) \), the marginal cost at that point.