10. (9 points) A piece of wire of length 40 cm is cut into two pieces. One piece is made into a circle; the rest is made into a square.

Find the lengths of each piece of wire so that the sum of the areas of the circle and square is a minimum. [Be sure to show *all* of your work and clearly identify your answers.]

Solution: *Let x denote the length of wire that is made into the circle, so that $40-x$ is the length of the piece of wire to be shaped into a square. The circumference of the circle is then equal to x so it must have radius $r = x/2\pi$. The square, made from the remaining piece of wire that has length $40-x$, then has side length $(40-x)/4$ so the area of the circle plus the area of the square is equal to*

$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{40-x}{16}\right)^2 = \frac{x^2}{4\pi} + \frac{(40-x)^2}{16}.$$

We see that the function $A(x)$ is a quadratic polynomial in x and the coefficient of x^2 is $(1/4\pi - 1/16)$ which is positive. Therefore, the graph of $A(x)$ is a parabola opening upward so it can have at most one critical point which will be a global minimum. The graph of $A(x)$ is shown in the figure below.

To find the critical point of A, we could either complete the square to write the quadratic function $A(x)$ in the form $a(x-b)^2+c$, or else find the zero of the derivative $A'(x)$. Here we will use the latter method and compute the solution of the linear equation $A'(x)=0$. That is,

$$0 = A'(x) = \frac{x}{2\pi} - \frac{40-x}{8} = x\left(\frac{1}{2\pi}+\frac{1}{8}\right) - 5 = x\left(\frac{8+2\pi}{16\pi}\right) - 5$$

or $\boxed{x = 40\pi/(4+\pi) \simeq 17.59603386cm}$. *This value of x is in the interval $0 < x < 40$ so it is the value which gives minimum area. The length of the other piece of wire is* $\boxed{40 - x = 160/(4+\pi) \simeq 22.40396614.}$ *The minimum value of the sum of the areas turns out to be approximately* $56.00991535\ cm^2$