9. (9 points) An ecologist is studying the biodiversity of an environment near the top edge of a windswept cliff. One statistic of interest to her is the distribution of biomass throughout the environment. If x measures the horizontal distance from the cliff edge in meters, there is only one species of tussock grass that grows for $1 \leq x \leq 20$. Along the first meter from the cliff's edge, nothing grows; and beyond 20 meters from the cliff, various other plant species thrive.

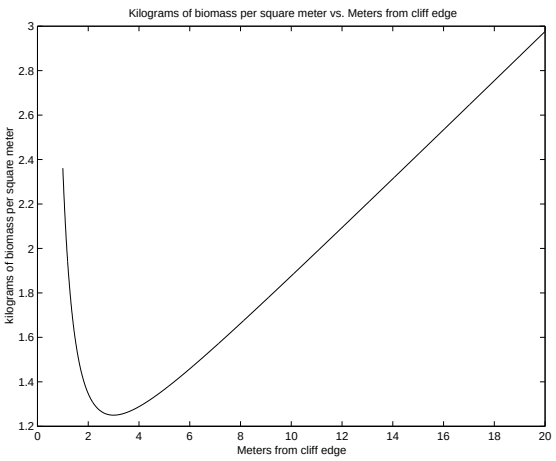
A typical tussock grass plant located x meters from the cliff edge has mass $\frac{2}{9}x^3 + \frac{3}{2}x^2 + 3$ kilograms per plant, and there will be $\frac{1}{2x^2}$ such plants per square meter.

(a) For $1 \leq x \leq 20$, find the distance from the cliff which minimizes the biomass per square meter. Show your work.

Solution: *The biomass B per square meter at a distance x meters from the cliff edge is*

$$B(x) = \{\#\ kg/plant\} \times \{\#\ plants/m^2\} = \left(\frac{2}{9}x^3 + \frac{3}{2}x^2 + 3\right) \times \left(\frac{1}{2x^2}\right) = \left(\frac{1}{9}x + \frac{3}{4} + \frac{3}{2x^2}\right)\ \ kg/m^2$$

We are asked to find the point x in the interval $1 \leq x \leq 20$ where $B(x)$ takes on its minimum value. Since B is given by a simple formula, we can look at a graph of B (for example, on a calculator) to get an idea of where the point x point might be. A graph of B is shown in the figure below for the relevant values of x and it appears that B has a minimum somewhere near the left end of this range. In particular, B should have one critical point in the interval which is a minimum.



To find the point x exactly, calculate: $dB/dx = \frac{1}{9} - \frac{3}{x^3}$. A critical point of B is a solution of the equation $dB/dx = 0$ or $\frac{1}{9} = \frac{3}{x^3}$; that is, where $x^3 = 27$. The only solution in the interval $1 < x < 20$ is $x = 3$. Since the graph shows that $B(x)$ is decreasing for $1 < x < 3$ and increasing for $3 < x < 20$, this must be a global minimum of B on the interval. The minimum value is $B(3) = (5/4)kg/m^2$.

At a distance of *3 meters* from the cliff's edge the biomass per square meter is minimized.

(b) What is the maximal biomass per square meter in this region? Explain.

Solution: *The maximum of $B(x)$ on the interval must occur either at a critical point or at an endpoint. The graph shows clearly that the maximum occurs at the right hand endpoint, or $x = 20$, so the maximal biomass is $B(20) = 21427/7200 \simeq 2.97597222$ or about $3kg/m^2$.*

The value of B at $x = 1$ is $B(1) = 85/36 \simeq 2.3611111kg/m^2$, so we see that both the values at the left hand endpoint and the critical point are smaller than the value at the right hand endpoint.

The maximal biomass per square meter in this region is *about $3kg/m^2$*.