8. (10 points) You have given up on your lemonade stand after your cousin ran it into the ground. However, you still need to make some money over the summer so you decide to tutor local high school students in mathematics. You start off charging $45 per hour. Only 2 students are willing to pay this rate for your expert knowledge. However, you find that for each $3 less per hour that you charge, 1 more student is willing to sign up for tutoring. You decide you can tutor for a maximum of 15 hours per week, that you will meet with each student one hour per week, and that you will only tutor one student at a time. What should you charge if making the most money per week is your only goal? In order to get full credit, you must use the techniques of calculus to solve this problem and show all of your work!

Let x be the number of additional students to the 2 you would have at $45 per hour. So the total number of students you have is $2 + x$ and the amount you charge each student is $45 - 3x$. The total amount of money you will make each week is $A(x) = (45 - 3x)(2 + x)$. To find out how much you should charge, you need to maximize $A(x)$. $A'(x) = 39 - 6x$, so setting this equal to zero we see we have a critical point at $x = 6.5$. $A''(6.5) = -6 < 0$ so this is a local maximum. However, we can't tutor half of a student, so we can either tutor 8 students or 9 students corresponding to $x = 6$ or $x = 7$. $A(6) = \$216$ and $A(7) = \$216$, so the best option would be to choose to $x = 6$ to make the same amount of money with less work. Now we need to check the endpoints of our interval which occur when you have 0 students or 15 students. If you have 0 students you clearly will make no money, so this can't be the maximum of $A(x)$ since our local maximum gives $216. The other endpoint gives $A(13) = \$90$. So our maximum does occur when $x = 6$. Therefore, we should charge $45 - 3(6) = \$27$ per hour.