7. (17 points) At the Wizard Fair, there is a booth where wizards win Bertie Bott's Every Flavor Beans. To determine how many beans one gets, a contestant is given a string 50 inches long. From this string, contestants can cut lengths to form an *equilateral triangle* and a *rectangle whose length is twice its width*. The number of Bertie Bott's beans one wins depends on the combined areas of the triangle and rectangle. Harry, knowing calculus, goes immediately to work setting up a function, finding critical points, etc.

(a) Use your knowledge of calculus to determine the areas of the triangle and rectangle that will maximize the number of beans that Harry can win. Show your work.

We need to find a formula for the total area of the triangle and rectangle. Let's begin by finding a formula for the area of the triangle. Say we cut the string x inches from the left end of the string, and suppose the left piece is used to make the equilateral triangle and the right piece is used to make the rectangle. So, each side of the triangle must have length $\frac{x}{3}$. An equilateral triangle has all of its angles equal to $\frac{\pi}{3}$. Let h be the height if the triangle. Then $sin(\frac{\pi}{3}) = \frac{h}{\frac{x}{3}}$ and $h = \frac{x\sqrt{3}}{6}$. So area of triangle $= (\frac{1}{2})(\frac{x}{3})(\frac{x\sqrt{3}}{6})$.

Now we need to find a formula for the area of the rectangle. Let the two sides of the rectangle be l and w. Then the perimeter of the rectangle must equal the length of the right piece of string. So, $2l + 2w = 50 - x$. We know though that the length is twice the width, which means $2(2w) + 2w = 6w = 50 - x$, and therefore, $w = \frac{50-x}{6}$. Since the area of the rectangle equals lw, we know that the area equals $2w^2 = 2\frac{(50-x)^2}{36}$.

$$\text{Thus we have } A_{total} = \frac{x^2\sqrt{3}}{36} + \frac{2(50-x)^2}{36}$$

$$= \frac{1}{36}((\sqrt{3}+2)^2 - 200x + 2(50)^2).$$

So $\frac{dA}{dt} = \frac{1}{36}(2(\sqrt{3}+2)x - 200)$. Setting the derivative equal to zero to find the critical point gives $0 = (\sqrt{3}+2)x - 100$ and thus $x \approx 26.79$. Notice though that $\frac{d^2A}{dt^2} = \frac{1}{18}(\sqrt{3}+2) > 0$, which means that A has a *minimum* at $x = 26.79$ and the maximum must occur at one of the endpoints - ie. when $x = 0$ or $x = 50$.

$x = 0$: $2w + 2l = 6w = 50 \Rightarrow w = 8.34$. Thus, $A_{total} = lw = 2(8.34)^2 = 136.89$ square inches.

$x = 50$: area $= \frac{1}{2}(\frac{50}{3})(\frac{50\sqrt{3}}{6}) \approx 120.28$ square inches.

This analysis says that Harry should not cut the string and the maximum occurs when the area of the triangle is zero and the area of the rectangle is 136.89 square inches.

(b) If the number of beans won is 9 times the combined area, what is the greatest number of beans a contestant can win?

This is just nine times the area we found in (a). So, the greatest number of beans a contestant can win is $9 * 136.89 = 1232$ beans.