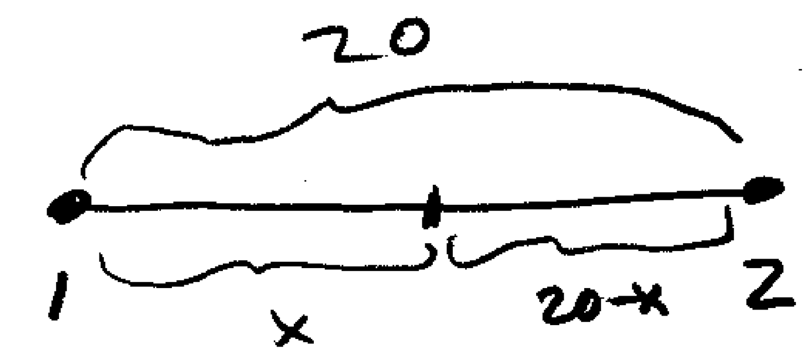
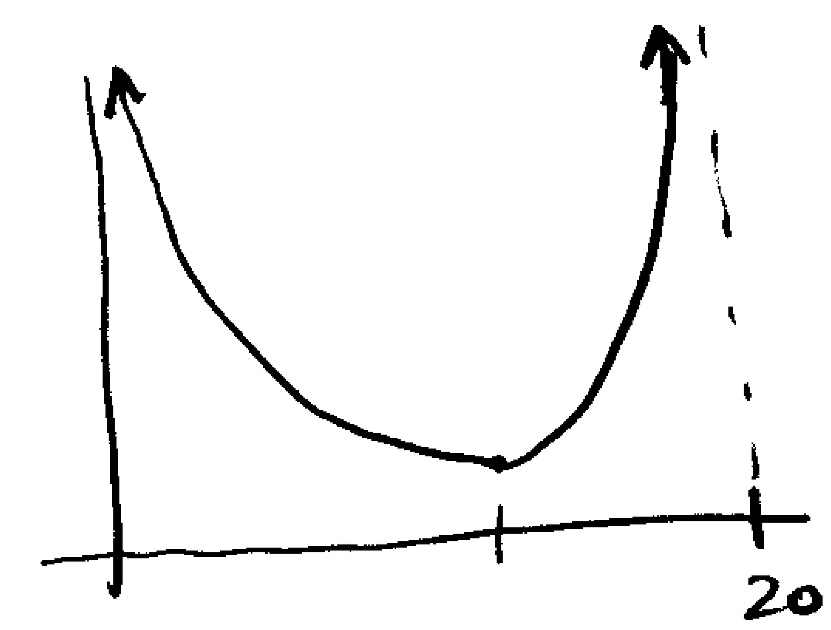
2. (10 points) A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}$$



where k_1 and k_2 are positive constants which depend on the quantity of smoke each stack is emitting. If $k_1 = 7k_2$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

Graph From Calculator



$$S = k_1 x^{-2} + k_2 (20-x)^{-2}$$

$$\frac{dS}{dx} = -2k_1 x^{-3} - 2k_2 (20-x)^{-3}(-1)$$

$$= 2\left(k_2(20-x)^{-3} - k_1 x^{-3}\right).$$

So

$$\frac{dS}{dx} = 0 \Rightarrow k_2(20-x)^{-3} = k_1 x^{-3}$$

$$\Rightarrow \frac{k_2}{k_1} = \frac{x^{-3}}{(20-x)^{-3}} = \left[\frac{x}{20-x}\right]^{-3} = \left[\frac{20-x}{x}\right]^3.$$

$$\Rightarrow \sqrt[3]{\frac{k_2}{k_1}} = \frac{20-x}{x} = \frac{20}{x} - 1$$

$$\Rightarrow 1 + \sqrt[3]{\frac{k_2}{k_1}} = \frac{20}{x} \Rightarrow x = \boxed{\frac{20}{1+\sqrt[3]{k_2/k_1}}}.$$

Check: 2nd derivative test.

$$\frac{d^2S}{dx^2} = 2\left[-3k_2(20-x)^{-4}(-1) - (-3k_1 x^{-4})\right]$$

$$= 6\left[\frac{k_2}{(20-x)^4} + \frac{k_1}{x^4}\right] > 0 \quad \text{Conc up} \Rightarrow \underline{\text{max}}.$$

If $k_1 = 7k_2$, smokestack 1 is 7 times as bad as the other.
And indeed, the min occurs at

$$x = \frac{20}{1+\sqrt[3]{1/7}} \approx \boxed{13.13},$$ closer to #2 than #1