2. (10 points) A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance \( x \) from one stack, is given by

\[
S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}
\]

where \( k_1 \) and \( k_2 \) are positive constants which depend on the quantity of smoke each stack is emitting. If \( k_1 = 7k_2 \), find the point on the line joining the stacks where the concentration of the deposit is a minimum.

\[
\frac{dS}{dx} = -2k_1 x^{-3} - 2k_2 (20-x)^{-3}(-1)
\]

\[
= 2 \left( k_2 (20-x)^{-3} - k_1 x^{-3} \right)
\]

So

\[
\frac{dS}{dx} = 0 \implies k_2 (20-x)^{-3} = k_1 x^{-3}
\]

\[
\implies \frac{k_2}{k_1} = \frac{x^{-3}}{(20-x)^{-3}} = \left[ \frac{x}{20-x} \right]^3 = \left[ \frac{20-x}{x} \right]^3
\]

\[
\implies 3\sqrt[3]{\frac{k_2}{k_1}} = \frac{20-x}{x} = \frac{20}{x} - 1
\]

\[
\implies 1 + 3\sqrt[3]{\frac{k_2}{k_1}} = \frac{20}{x} \implies x = \frac{20}{1 + 3\sqrt[3]{\frac{k_2}{k_1}}}
\]

Check: 2nd derivative test

\[
\frac{d^2S}{dx^2} = 2 \left[ -3k_2 (20-x)^{-4}(-1) - (-3k_1 x^{-4}) \right]
\]

\[
= 6 \left[ \frac{k_2}{(20-x)^4} + \frac{k_1}{x^4} \right] > 0 \quad \text{conc up} \implies \max.
\]

If \( k_1 = 7k_2 \), smokestack 1 is 7 times as bad as the other. And indeed, the min occurs at

\[
x = \frac{20}{1 + 3\sqrt[3]{1/7}} = 13.13\quad \text{closer to #2 than #1}
\]