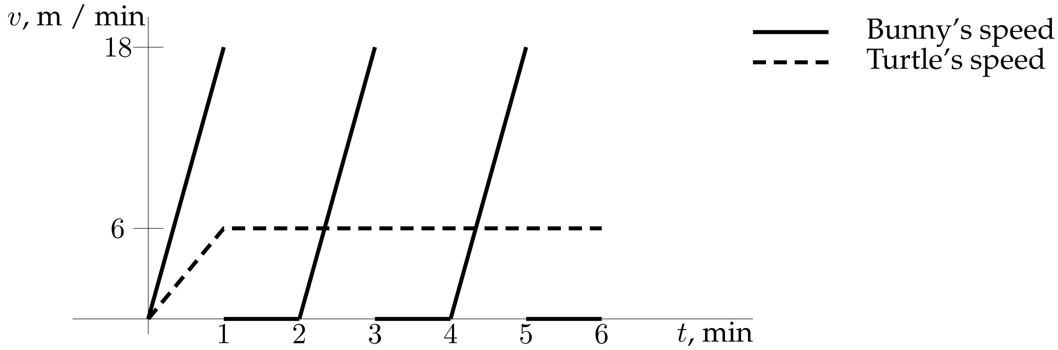
6. The Awkward Turtle is competing in a race! Unfortunately his archnemesis, the Playful Bunny, is also in the running. The two employ very different approaches: the Awkward Turtle takes the first minute to accelerate to a slow and steady pace which he maintains through the remainder of the race, while the Playful Bunny spends the first minute accelerating to faster and faster speeds until she's exhausted and has to stop and rest for a minute - and then she repeats this process until the race is over. The graph below shows their speeds (in meters per minute), t minutes into the race. (Assume that the pattern shown continues for the duration of the race.)



(a) (6 points) What is the Awkward Turtle's average speed over the first two minutes of the race? What is the Playful Bunny's?

Let $T(t)$ denote the Awkward Turtle's velocity t minutes into the race, and $B(t)$ the Playful Bunny's. Then the Awkward Turtle's average velocity over the first two minutes is

$$\frac{1}{2}\int_0^2 T(t)\,dt = \frac{1}{2}\left(\int_0^1 T(t)\,dt + \int_1^2 T(t)\,dt\right) = \frac{1}{2}(3+6) = 4.5 \text{ m / min}$$

Similarly, the Playful Bunny's average velocity over the first two minutes is

$$\frac{1}{2}\int_0^2 B(t)\,dt = \frac{1}{2}\left(\int_0^1 B(t)\,dt + \int_1^2 B(t)\,dt\right) = \frac{1}{2}(9+0) = 4.5 \text{ m / min}$$

(b) (3 points) The Playful Bunny immediately gets ahead of the Awkward Turtle at the start of the race. How many minutes into the race does the Awkward Turtle catch up to the Playful Bunny for the first time? Justify your answer.

From part (a), we know that at two minutes into the race, the Awkward Turtle and the Playful Bunny have run the exact same distance. So, it remains to determine whether this is the first time the Turtle catches up. Since the Bunny is running faster than the Turtle the entire first minute of the race, it's clear that the Bunny is ahead that whole time, so the Turtle couldn't have caught up during the first minute. During the second minute of the race, the Bunny is standing still, so the first time the Turtle catches up must be precisely two minutes into the race.

(c) (5 points) If the race is 60 meters total, who wins? Justify your answer.

The distance the Turtle has run after x minutes is

$$\int_0^x T(t)\,dt = \int_0^1 T(t)\,dt + \int_1^x T(t)\,dt = 3 + 6(x-1) = 6x - 3 \text{ meters.}$$

Therefore it takes the Awkward Turtle 10.5 minutes to finish the race. By contrast, after 10.5 minutes the Playful Bunny has run

$$\begin{aligned}\int_0^{10.5} B(t)\,dt &= \int_0^1 B(t)\,dt + \int_2^3 B(t)\,dt + \cdots + \int_8^9 B(t)\,dt + \int_{10}^{10.5} B(t)\,dt \\ &= (5)(9) + 2.25 \\ &= 47.25 \text{ meters.}\end{aligned}$$

So the Awkward Turtle wins!