8. (10 points) A typical student spends the 24 hours leading up to this exam sleeping, studying, eating, and Facebook stalking. Suppose the total amount of time spent on eating and Facebook is 8 hours. The student's score on the exam, E (out a possible 100 points) depends on S, the number of hours of sleep the student enjoys during the 24 hours leading up to the exam. To be precise,

$$E(S) = 40\sin\left(\frac{5\pi}{51}(S-3.4)\right) + 36$$

How many hours should the student study in the day leading up to the exam to maximize his / her score?

[*You must use calculus - not just your calculator - and show your work to receive full credit.*]

We are looking for the global maximum of E. Thus start by identifying which values of S make $E'(S) = 0$:

$$E'(S) = (40)\left(\frac{5\pi}{51}\right)\cos\left(\frac{5\pi}{51}(S-3.4)\right) = 0$$

implies

$$\frac{5\pi}{51}(S-3.4) = \frac{\pi}{2} + k\pi$$

for some integer k. It follows that any solution to this is of the form $S = 8.5 + \frac{51k}{5}$ for k an integer. Since taking any negative value of k gives a negative value of S, and taking any positive value of k gives a value of S larger than 16, the only solution to $E'(S) = 0$ is $S = 8.5$ hours. Also, since $E'(S) > 0$ for S slightly below 8.5 and $E'(S) < 0$ for S slightly larger than 8.5, E must attain a maximum at $S = 8.5$.

To check that this is the global maximum on the interval, it suffices to test the the endpoints. Plugging these in, we find $E(0) \approx 1.359$ and $E(16) \approx 9.052$, both of which are smaller than $E(8.5) = 76$. So, the student's exam score is maximized when he / she sleeps 8.5 hours; this means the student must study $16 - 8.5 = 7.5$ hours to maximize his / her score.