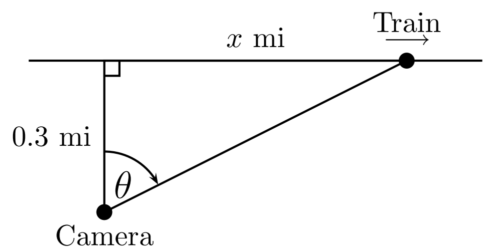
8. [12 points]
A train is traveling eastward at a speed of 0.4 miles per minute along a long straight track, and a video camera is stationed 0.3 miles from the track, as shown in the figure. The camera stays in place, but it rotates to focus on the train as it moves.
Suppose that t is the number of minutes that have passed since the train was directly north of the camera; after t minutes, the train has moved x miles to the east, and the camera has rotated θ radians from its original position.



a. [3 points] Write an equation that expresses the relationship between x and θ.

Solution:

$$\tan(\theta) = \frac{x}{0.3}, \text{ or } \theta = \arctan\left(\frac{x}{0.3}\right)$$

b. [4 points] Suppose that seven minutes have passed since the train was directly north of the camera. How far has the train moved in this time, and how much has the camera rotated?

Solution: The train's velocity is constant (i.e. $\frac{dx}{dt} = 0.4$), so we can use the formula $\textit{distance} = \textit{velocity} \cdot \textit{time}$. Thus, the train has moved $\left(0.4\frac{\text{mi}}{\text{min}}\right)(7\text{min}) = 2.8$ miles.

Using the fact that $x = 2.8$, we have $\theta = \tan^{-1}(\frac{2.8}{0.3}) \approx 1.4641$, so the camera has rotated 1.4641 radians in the clockwise direction.

c. [5 points] How fast is the camera rotating (in radians per minute) when $t = 7$?

Solution: We want to find $\frac{d\theta}{dt}$. To do so, we will (implicitly) take the derivative of our equation from part (a), with respect to t.

$$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\left(\frac{x}{0.3}\right), \text{ or } \frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{1}{0.3}\frac{dx}{dt}$$

We can then plug in $\frac{dx}{dt} = 0.4$ and $\theta = 1.4641$:

$$\frac{1}{\cos^2(1.4641)}\frac{d\theta}{dt} = \frac{0.4}{0.3}, \text{ so } \frac{d\theta}{dt} = \frac{0.4}{0.3}\cos^2(1.4641) \approx 0.01513.$$

Thus, the camera is rotating at a speed of 0.01513 radians per minute (in the clockwise direction) when $t = 7$.