1. [10 points] Find a formula for a function of the form

$$f(x) = \frac{1}{a + x + bx^2}$$

which has a local minimum at $(2, 1/2)$. Be sure to show that your function has a minimum at $(2, 1/2)$.

Solution: A local minimum will occur when $f'(x) = 0$ or is undefined. The former is when

$$-\frac{2bx + 1}{(bx^2 + x + a)^2} = 0.$$

The numerator is zero when $x = -\frac{1}{2b}$, and the denominator is zero on either side of this value, when $x = -\frac{1}{2b} \pm \frac{1}{2b}\sqrt{1 - 4ab}$. The first of these gives the local minimum we want: if we let $-\frac{1}{2b} = 2$, we have $b = -\frac{1}{4}$. This gives $f'(x) = -\frac{-\frac{1}{2}x+1}{(-\frac{1}{4}x^2+x+a)^2}$, so that if $x < 2$ we have $f'(x) < 0$, and if $x > 2$, $f'(x) > 0$. Thus if $b = -\frac{1}{4}$, $x = 2$ is a local minimum. To require that the minimum occur at $(2, 1/2)$, we want $f(2) = \frac{1}{a+2-1} = 1/2$, so that $a = 1$.