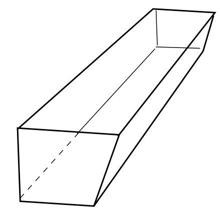
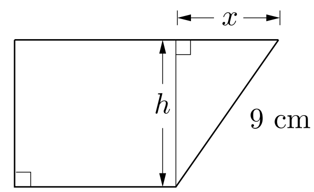
2. [13 points] A rain gutter attaches to the edge of a roof and collects the rain that falls on the roof. A common gutter design is shown in the figure to the right, and has a trapezoidal cross-section (also shown). In this problem we consider a gutter with base and side length 9 cm, as shown.





a. [1 point] Write an equation which relates the length x to the height h.

Solution: Using the pythagorean theorem, we have $h^2 = 81 - x^2$.

b. [4 points] Using your equation from (a), write an equation for the cross-sectional area of the gutter as a function of the length x (note that the area is the sum of a rectangular and right-triangular region).

Solution: We have $A(x) = 9h + \frac{1}{2}\,x\,h = 9\sqrt{81 - x^2} + \frac{1}{2}\,x\,\sqrt{81 - x^2}$.

c. [8 points] Find the length x that gives the maximum cross-sectional area. Be sure to show work that demonstrates that you have found the maximum.

Solution: Note that the domain of interest for our area function is $0 \le x < 9$: taking $x < 0$ clearly gives a smaller cross-sectional area than $x > 0$, and $x = 9$ is clearly the largest value x can take (and the cross-sectional area goes to zero for $x = 9$). To find the maximum we first locate critical points. These are where $A'(x) = 0$ or is undefined. This gives

$$A'(x) = -\frac{9x}{\sqrt{81 - x^2}} + \frac{1}{2}\sqrt{81 - x^2} - \frac{1}{2}\frac{x^2}{\sqrt{81 - x^2}} = 0.$$

The value $x = 9$ lies outside our domain, so we need not worry about the derivative being undefined there. Multiplying through by $\sqrt{81 - x^2}$, we have $-x^2 - 9x + \frac{81}{2} = 0$. Thus, using the quadratic formula, critical points are where $x = -\frac{9}{2} \pm \frac{1}{2}\sqrt{3(81)}$.
We want a positive value of x, so take $x = x_c = \frac{9}{2}\,(\sqrt{3} - 1) \approx 3.294$. We can see from the derivative (which is a downward opening parabola divided by a positive square root) that this positive critical point must have A' going from positive to negative, and is thus a maximum. As the only critical point on the domain, $x = x_c$ must also be the global maximum.
Alternately, we can check the sign of A' on either side of x_c: $A'(0) = 9/2 > 0$ and $A'(4) \approx -1.43 < 0$. The second derivative test will also work, but the evaluation is more difficult.