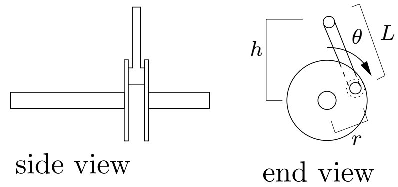
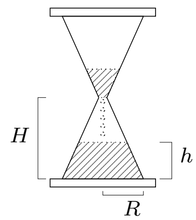
3. [10 points] For each of the following determine the indicated quantity.

Crank shaft diagram (part a)



a. [4 points] In an internal combustion engine, pistons are pushed up and down by a crank shaft similar to the diagram shown to the right. As the shaft rotates the height of the piston, h, is related to the rotational angle θ of the shaft by $h = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where r and L are constant lengths. If $r = 10$ cm, $L = 15$ cm, and h is decreasing at a rate of 5000 cm/s when $\theta = 3\pi/4$, how fast is θ changing then?

Hourglass diagram (part b)



Solution: Using the chain rule, we know that $h'(t) = \left(-r\sin\theta - \frac{r^2\sin\theta\cos\theta}{\sqrt{L^2-r^2\sin^2\theta}}\right)\cdot\frac{d\theta}{dt}$. Thus if $\theta = 3\pi/4$, $h' = -5000$, $r = 10$ and $L = 15$, we have $-5000 = \left(-\frac{10}{\sqrt{2}} - \frac{50}{\sqrt{225-50}}\right)\cdot\frac{d\theta}{dt} \approx -3.29\cdot\frac{d\theta}{dt}$. Thus $\frac{d\theta}{dt} \approx 1500$ radians/sec

b. [6 points] The lower chamber of an hourglass is shaped like a cone with height H in and base radius R in, as shown in the figure to the right, above. Sand falls into this cone. Write an expression for the volume of the sand in the lower chamber when the height of the sand there is h in *(Hint: A cone with base radius r and height y has volume $V = \frac{1}{3}\pi r^2 y$, and it may be helpful to think of a difference between two conical volumes.)*. Then, if $R = 0.9$ in, $H = 2.7$ in, and sand is falling into the lower chamber at 2 in^3/min, how fast is the height of the sand in the lower chamber changing when $h = 1$ in?

Solution: The whole volume of the lower chamber is $V_{tot} = \frac{1}{3}\pi R^2 H$. The volume of the empty space above the sand is similarly $V_{emp} = \frac{1}{3}\pi r^2 (H-h)$, where r is the radius at the height h. By comparing the similar triangles delimiting the full lower chamber and the empty top section, we see that $r = \frac{R}{H}(H-h)$. Thus $V_{emp} = \frac{1}{3}\pi\frac{R^2}{H^2}(H-h)^3$. The volume of the lower, sand-filled region is therefore

$$V = \frac{1}{3}\pi\frac{R^2}{H^2}\left(H^3 - (H-h)^3\right).$$

Then, differentiating, we have

$$\frac{dV}{dt} = \pi\frac{R^2}{H^2}(H-h)^2\frac{dh}{dt}.$$

Thus, when $\frac{dV}{dt} = 2$, $R = 0.9$, $H = 2.7$, and $h = 1$,

$$2 = \pi\frac{1}{3^2}(1.7)^2\frac{dh}{dt},$$

so that $\frac{dh}{dt} = \frac{18}{\pi(1.7)^2} \approx 1.98$ in/min.