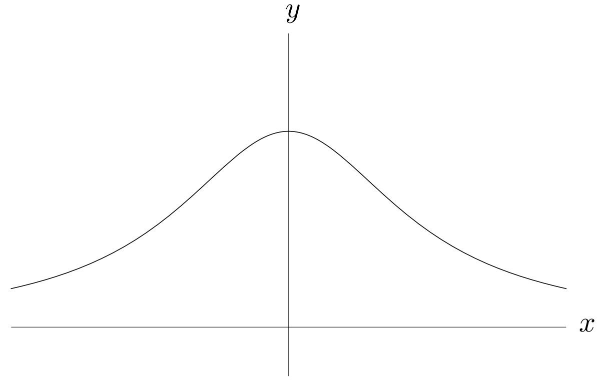
5. [12 points] A rectangle has one side on the x-axis and two vertices on the curve

$$y = \frac{36}{9 + x^2}.$$

This curve is graphed below. Find the x- and y- coordinates of all four vertices of the rectangle with largest area. You must show that your vertices maximize the area of the rectangle.



Solution: The area of such a rectangle, if its lower-right corner is at $(x, 0)$, is

$$A = 2x\left(\frac{36}{9 + x^2}\right) = \frac{72x}{9 + x^2}.$$

Differentiating this gives

$$\frac{dA}{dx} = \frac{(9 + x^2)72 - 72x(2x)}{(9 + x^2)^2} = \frac{648 - 72x^2}{(9 + x^2)^2},$$

so there is a critical point where the numerator is zero, which is

$$x = 3.$$

We can verify using the first derivative test that this is a local maximum (for example, by noticing that dA/dx is positive at $x = 0$ and negative at $x = 4$), and since it is the only critical point on the domain $(0, \infty)$ we are considering, it must be the global maximum. Plugging into the equation for the curve shows that $y = 2$ when $x = 3$. Therefore, the four vertices of the rectangle are

$$(\pm 3, 2), (\pm 3, 0).$$