8. [13 points] Use the family of functions of the form

$$f(x) = ax - \ln(1 + e^{bx})$$

to answer the following questions. The constants a and b are both positive.

a. [4 points] Use the given formula for $f(x)$ to give an explicit expression for the limit definition of $f'(x)$. Check your expression carefully, as **no partial credit** will be given on this part of the problem. Do not evaluate your expression.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{a(x+h) - \ln(1 + e^{b(x+h)}) - ax + \ln(1 + e^{bx})}{h}.$$

b. [4 points] Compute $f'(x)$ using the rules of differentiation. Do not try to evaluate your expression from (a).

Solution:

$$f'(x) = a - \frac{be^{bx}}{1 + e^{bx}}.$$

c. [5 points] When $a < b$, the function $f(x)$ has a critical point at

$$x = \frac{1}{b} \ln\left(\frac{a}{b-a}\right).$$

Using the **second-derivative test**, determine whether this critical point is a local maximum, local minimum, or neither.

Solution: The second derivative is

$$f''(x) = -\frac{(1 + e^{bx})b^2 e^{bx} - be^{bx}be^{bx}}{(1 + e^{bx})^2} = -\frac{b^2 e^{bx}}{(1 + e^{bx})^2}.$$

This is always negative, so the critical point is a local maximum.