3. [11 points] For positive constants a and b, the potential energy of a particle is given by

$$U(x) = a\left(\frac{5b^2}{x^2} - \frac{3b}{x}\right).$$

Assume that the domain of $U(x)$ is the interval $(0, \infty)$.

a. [2 points] Find the asymptotes of $U(x)$. If there are none of a particular type, write NONE.

Solution: We can get a common denominator and write

$$U(x) = a\frac{5b^2 - 3bx}{x^2}.$$

We see that there is a vertical asymptote at $x = 0$, where the denominator is zero, and a horizontal asymptote at $U = 0$, since the degree of the denominator is greater than the degree of the numerator.

Answer: Vertical asymptote(s): $x = 0$ Horizontal asymptote(s): $U = 0$

b. [6 points] Find the x-coordinates of all local maxima and minima of $U(x)$ in the domain $(0, \infty)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Solution: First we find critical points by looking at where $U'(x)$ is undefined or zero. We have

$$U'(x) = a\left(-\frac{10b^2}{x^3} + \frac{3b}{x^2}\right) = a\frac{3bx - 10b^2}{x^3}.$$

There are no points in the domain of $U(x)$ where $U'(x)$ is undefined, but $U'(x)$ has a zero where $3bx - 10b^2 = 0$, or $x = \frac{10b}{3}$.
To classify this critical point, we can use the First or Second Derivative Test. We will use the Second Derivative Test here, so we compute

$$U''(x) = a\left(\frac{30b^2}{x^4} - \frac{6b}{x^3}\right) = \frac{6ab}{x^4}(5b - x).$$

Since a, b, and x^4 are always positive and $5b - x$ is positive at $x = \frac{10b}{3}$, we see that $U''\left(\frac{10b}{3}\right) > 0$, and hence $x = \frac{10b}{3}$ is a local minimum.
There are no other critical poitns to consider, so there are no local maxima.

Answer: Local max(es) at $x =$ NONE Local min(s) at $x =$ $\frac{10b}{3}$

c. [3 points] Suppose $U(x)$ has an inflection point at $(6, -14)$. Find the values of a and b. *Show your work, but you do not need to verify that this point is an inflection point.*

Solution: We already found $U''(x) = \frac{6ab}{x^4}(5b - x)$, so we see that the only potential inflection point occurs at $x = 5b$, the only place in the domain of $U(x)$ where $U''(x)$ is zero or undefined. Hence $5b = 6$ or $b = 1.2$.
Plugging in $x = 6$, $b = 1.2$, and $U = -14$ into the original equation for $U(x)$ yields

$$-14 = a\left(\frac{1}{5} - \frac{3}{5}\right) = -\frac{2a}{5}$$

and hence $a = 35$.

Answer: $a =$ 35 and $b =$ 1.2