2. [13 points] For nonzero constants a and b with $b > 0$, consider the family of functions given by

$$f(x) = e^{ax} - bx\,.$$

Note that the derivative and second derivative of $f(x)$ are given by

$$f'(x) = ae^{ax} - b \qquad \text{and} \qquad f''(x) = a^2 e^{ax}\,.$$

a. [6 points] Suppose the values of a and b are such that $f(x)$ has at least one critical point. For the domain $(-\infty, \infty)$, find all critical points of $f(x)$, all values of x at which $f(x)$ has a local extremum, and all values of x at which $f(x)$ has an inflection point.
Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema and inflection points.
(Note that your answer(s) may involve the constants a and/or b.)

Solution: To find critical points we (i) solve for x in the equation $f'(x) = 0$ and (ii) look for points in the domain of the function where the derivative is undefined. Solving $f'(x) = 0$ we have $ae^{ax} - b = 0$ so

$$e^{ax} = \frac{b}{a} \tag{1}$$

and we see that there is a critical point at $x = \frac{\ln(\frac{b}{a})}{a}$ (as long as $b/a > 0$).
The derivative is defined everywhere so this is the only critical point. (Since we are to assume there is at least one critical point, note that this implies that b/a must be positive.) To classify this critical point we can use the second derivative test. Since $a \neq 0$, we have that $f''(x) > 0$ for all x, so the function $f(x)$ is always concave up. Therefore, the critical point at $x = \frac{\ln(\frac{b}{a})}{a}$ is a local min.
To find points of inflection we (i) solve for x in the equation $f''(x) = 0$, (ii) look for any points in the domain of the function where the second derivative is undefined and (iii) check that $f''(x)$ changes sign at any points found. For the function $f(x)$, $f''(x)$ is never 0 and is defined for all x so we have no points of inflection.

(For each answer blank below, write NONE in the answer blank if appropriate.)

critical point(s) at $x =$ $x = \dfrac{\ln(\frac{b}{a})}{a}$ local min(s) at $x =$ $x = \dfrac{\ln(\frac{b}{a})}{a}$

inflection point(s) at $x =$ **NONE** local max(es) at $x =$ **NONE**

b. [2 points] Which of the following conditions on the constant a guarantee(s) that $f(x)$ has at least one critical point in its domain $(-\infty, \infty)$? Circle all the cases in which $f(x)$ definitely has at least one critical point. *Hint: There is at least one such condition listed.*

i. $a < 0$ ii. $\boxed{0 < a < b}$ iii. $\boxed{b < a}$

Solution: When finding the critical point in part (a), we had to solve equation (1), which only has a solution if $\frac{b}{a} > 0$. Since $b > 0$, this means that $f(x)$ has a critical point if and only if $a > 0$, which is true both if $0 < a < b$ and if $b < a$ (since $b > 0$).

c. [5 points] Find exact values of a and b so that $f(x)$ has a critical point at $(1, 0)$. Remember to show your work carefully.

Solution: First, $(1, 0)$ must lie on the curve so we get the equation $e^a - b = 0$ which implies $b = e^a$. Next, $x = 1$ must be a critical point so $ae^a - b = 0$. Substituting $b = e^a$ into the second equation we get $ae^a - e^a = 0$ which implies $a = 1$. So $a = 1$ and $b = e$.

Answer: $a =$ **1** and $b =$ **e**