4. [12 points]
Having taken care of Sebastian and sent Erin into the hands of the Illumisqati, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.

In case they are helpful, recall the following formulas for a cone of radius $r$ and height $h$:

Volume = $\frac{1}{3} \pi r^2 h$ and Surface Area = $\pi r (r + \sqrt{h^2 + r^2})$.

a. [6 points] Let $g$ be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of $g$ when Roderick has poured a total of 20,000 mm$^3$ of chocolate into the mould? Show your work carefully, and make sure your answer is accurate to at least two decimal places.

**Solution:** The volume of the solid is given by $V = \frac{1}{3} \pi (20)^2 (60) - \frac{1}{3} \pi r^2 (60 - g)$ where $r$ is the radius of the cross-section at height $g$. We want to rewrite $r$ in terms of $g$. Using similar triangles we find the equation $r = \frac{20}{60 - g}$, which implies $r = \frac{60 - g}{3}$. Therefore, $V = 8000 \pi - \frac{\pi}{27} (60 - g)^3$. So, to find the appropriate $g$ we need to solve $8000 \pi - \frac{\pi}{27} (60 - g)^3 = 20,000$. Solving, we get

$$(60 - g)^3 = \frac{27}{\pi} (8000 \pi - 20,000),$$

which implies $g = 60 - \sqrt[3]{\frac{27}{\pi} (8000 \pi - 20,000)} \approx 24.67$. The chocolate is approximately 24.67 mm deep when he has poured a total of 20,000 mm$^3$ of chocolate into the mould.

**Answer:** $g \approx 24.67$

b. [6 points] How fast is the depth of the chocolate in the mould ($g$ in the diagram above) changing when Roderick has already poured 20,000 mm$^3$ of chocolate into the mould if he is pouring at a rate of 5,000 mm$^3$ per second at this time? Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.

**Solution:** We want to find $\frac{dg}{dt}$ when $g = 60 - \sqrt[3]{\frac{27}{\pi} (8000 \pi - 20,000)}$ (from part (a)) if $\frac{dV}{dt} = 5000$ at this time. Differentiating our formula (2) from part (a) with respect to $t$, we have

$$\frac{dV}{dt} = \frac{\pi}{9} (60 - g)^2 \frac{dg}{dt}.$$  

Substituting $g = 60 - \sqrt[3]{\frac{27}{\pi} (8000 \pi - 20,000)}$ and $\frac{dV}{dt} = 5000$ into this equation, we find

$$5000 = \frac{\pi}{9} \left(60 - \sqrt[3]{\frac{27}{\pi} (8000 \pi - 20,000)}\right)^2 \frac{dg}{dt}$$

so

$$\frac{dg}{dt} = \frac{5000 \cdot 9}{\pi} \left(\frac{27}{\pi} (8000 \pi - 20,000)\right)^{-2/3} \approx 11.47.$$  

(Using our approximation $g \approx 24.67$ instead gives us $\frac{dg}{dt} \approx 11.48$.)

So the depth of the chocolate is increasing at an instantaneous rate of about 11.47 mm/sec at that moment.

**Answer:** $11.47 \text{ mm/sec}$.