5. [8 points] Reggie is starting a fruit punch company. He has determined that the total cost, in dollars, for him to produce q gallons of fruit punch can be modeled by

$$C(q) = 100 + q + 25e^{q/100}.$$

Reggie can sell up to 100 gallons to Chris at a price of \$4 per gallon, and he can sell the rest to Alice at a price of \$3 per gallon. Assume that Reggie sells all of the fruit punch that he produces.

Note: Assume that the quantities of fruit punch produced and sold do not have to be whole numbers of gallons. (For example, Reggie could produce exactly $50\sqrt{2}$ gallons of fruit punch and sell all of these to Chris, who would pay a total of $200\sqrt{2}$ dollars for them.)

a. [4 points] For what quantities of fruit punch sold would Reggie's marginal revenue equal his marginal cost?

Solution: Reggie's marginal cost is $MC = C'(q) = 1 + \frac{1}{4}e^{q/100}$ and his marginal revenue is $MR = \begin{cases} 4 & \text{if } 0 < q < 100 \\ 3 & \text{if } 100 < q. \end{cases}$

So we solve $MR = MC$ separately for the two intervals $0 < q < 100$ and $q > 100$.

For $0 < q < 100$:

$$\begin{aligned} 1 + \frac{1}{4}e^{q/100} &= 4 \\ \frac{1}{4}e^{q/100} &= 3 \\ e^{q/100} &= 12 \\ q = 100\ln(12) &\approx 248.49. \end{aligned}$$

So marginal cost does not equal marginal revenue anywhere on the interval $0 < q < 100$ (because $100\ln(12) > 100$).

For $q > 100$:

$$\begin{aligned} 1 + \frac{1}{4}e^{q/100} &= 3 \\ \frac{1}{4}e^{q/100} &= 2 \\ e^{q/100} &= 8 \\ q = 100\ln(8) &\approx 207.94 \end{aligned}$$

Hence, marginal revenue equals marginal cost at $q = 100\ln(8)$.

Answer: $100\ln(8) \approx 207.94$ **gallons**

b. [4 points] Assuming that Reggie can produce at most 200 gallons of fruit punch, how much fruit punch should he produce in order to maximize his profit, and what would that maximum profit be? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: First, we find all critical points of the profit function $\pi(q)$ in the interval $0 \leq q \leq 200$. In part **a.**, we found that $\pi'(q) = 0$ only at $q \approx 207.94$, which is not in the interval $[0, 200]$. The other critical points of $\pi(q)$ occur where $\pi'(q)$ is not defined, namely, at $q = 100$.

Note that Reggie's revenue is a continuous function of q. So $\pi(q)$ is continuous on the interval $[0, 200]$ and we can apply the Extreme Value Theorem. It therefore suffices to compare the value of $\pi(q)$ at the endpoints ($q = 0$ and $q = 200$) and at the critical point ($q = 100$):

$$\begin{aligned} \pi(0) &= 0 - (100 + 0 + 25e^0) = -125 \\ \pi(100) &= 4(100) - (100 + 100 + 25e^1) \approx 132.04 \\ \pi(200) &= 4(100) + 3(100) - (100 + 200 + 25e^2) \approx 215.27 \end{aligned}$$

Hence, Reggie should produce 200 gallons of fruit punch for a profit of about \$215.27.

Answer: gallons of fruit punch: 200 and max profit: \$215.27