7. [9 points] Consider the family of functions given by $f(x) = e^{x^2+Ax+B}$ for constants A and B.

a. [6 points] Find and classify all local extrema of $f(x) = e^{x^2+Ax+B}$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write NONE if appropriate. Your answers may depend on A and/or B.

Solution: First, we find the critical points of $f(x)$. Notice that $f(x)$ is differentiable, so $f(x)$ only has critical points where $f'(x) = 0$. Since

$$f'(x) = (2x + A)e^{x^2+Ax+B},$$

the only critical point of $f(x)$ occurs where $2x+A = 0$, i.e., at $x = -\frac{A}{2}$. We test whether this critical point is a local maximum, a local minimum, or neither.
Applying the First Derivative Test:

- For $x < -\frac{A}{2}$: $2x + A < 0$ and $e^{x^2+Ax+B} > 0$, so $f'(x) < 0$.
- For $x > -\frac{A}{2}$: $2x + A > 0$ and $e^{x^2+Ax+B} > 0$, so $f'(x) > 0$.

Hence, $f'(x)$ changes from negative to positive at $x = -\frac{A}{2}$, so $f(x)$ has a local minimum at $x = -\frac{A}{2}$ (and no local maxima).
(Note that we could instead apply the Second Derivative Test: $f''(x) = (2x + A)(2x + A)(e^{x^2+Ax+B} + 2(e^{x^2+Ax+B}) = \left((2x + A)^2 + 2\right) e^{x^2+Ax+B}$ which is always positive (since both factors are always positive). So in particular $f''(-A/2) > 0$ so $f(x)$ has a local minimum at $x = -A/2$.

Answer: Local min(s) at $x =$ $-\frac{A}{2}$

Answer: Local max(es) at $x =$ NONE

b. [3 points] Find exact values of the constants A and B so that the point $(3, 1)$ is a critical point of $f(x) = e^{x^2+Ax+B}$.

Solution: As we showed in part **a.**, $f(x)$ has its only critical point at $x = -\frac{A}{2}$, so now we must have that $-\frac{A}{2} = 3$ so $A = -6$.
To find B, we use the fact that $(3, 1)$ is a point on the graph of $y = f(x)$ (so $f(3) = 1$).

$$\begin{aligned} 1 &= f(3) = e^{3^2+(-6)(3)+B} = e^{B-9} \\ \ln(1) &= \ln\left(e^{B-8}\right) \\ 0 &= B - 9 \\ B &= 9 \end{aligned}$$

Hence, $A = -6$ and $B = 9$.

Answer: $A =$ -6 and $B =$ 9