7. [9 points] Consider the family of functions

$$f(x) = ax^2e^{-bx}$$

where a and b are positive constants. Note that

$$f'(x) = ax(2 - bx)e^{-bx}.$$

a. [4 points] Find the exact values of a and b so that $f(x)$ has a critical point at $(4, e^{-2})$.

Solution: Since $(4, e^{-2})$ is a critical point of $f(x)$, then $f'(4) = 0$ or $0 = 4a(2 - 4b)e^{-4b}$. From this equation we get that $2 - 4b = 0$ (since $a, e^{-4b} > 0$). Then $b = 0.5$. We also know that the point $(4, e^{-2})$ is in the graph of $y = f(x)$, then $e^{-2} = 16ae^{-4b}$. Plugging the value of b, we get $e^{-2} = 16ae^{-2}$. This yields $1 = 16a$, so $a = \frac{1}{16}$.

b. [5 points] Using your values of a and b from the previous part, find and classify the local extrema of $f(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence that you have found them all. For each answer blank, write NONE if appropriate.

Solution: With the values found above $f'(x) = \frac{1}{16}x(2 - 0.5x)e^{-0.5x}$. The critical points are found by solving

$$f'(x) = \frac{1}{16}x(2 - 0.5x)e^{-0.5x} = 0$$

In this case, we have $x = 0$ or $2 - 0.5x = 0$ (since $.e^{-0.5x} > 0$). Hence the critical points are $x = 0$ and $x = 4$. To classify them we use the first derivative test:

- $f'(-1) = \frac{1}{16}(-1)(2 - 0.5(-1))e^{-0.5(-1)} = -0.257$
- $f'(1) = \frac{1}{16}(2 - 0.5)e^{-0.5} = 0.0568$
- $f'(5) = \frac{1}{16}(5)(2 - 0.5(5))e^{-0.5(5)} = -0.0128$

OR

- $f'(-1) = (-)(+)(+) = -$
- $f'(1) = (+)(+)(+) = +$
- $f'(5) = (+)(-)(+) = -$

Answer: Local max(es) at $x = 4$ Local min(s) at $x = 0$

© 2017 Univ. of Michigan Dept. of Mathematics (Creative Commons By−NC−SA license)