9. [9 points] A Math 115 coordinator is trying to create functions with certain properties in order to test students' understanding of various calculus concepts.

a. [5 points] He wants a function $f(x)$ of the form

$$f(x) = \begin{cases} ax^2 + ax + be^x & \text{for } x < 0 \\ a + 2\cos(x) & \text{for } x \geq 0 \end{cases}$$

where a and b are constants.

Find all value(s) of a and b for which $f(x)$ be differentiable at $x = 0$. Show enough work to justify your answer.

Solution: In order for $f(x)$ to be differentiable at $x = 0$, $f(x)$ has to be continuous. Then

$$\lim_{x\to 0^-} ax^2 + ax + be^x = b = \lim_{x\to 0^+} a + 2\cos(x) = a + 2 = f(0).$$

Then $b = a + 2$. If $f(x)$ is differentiable then

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = 2a(0) + a + be^0 = a + b.$$

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = -2\sin(0) = 0.$$

Hence $a + b = 0$. Using both equation we obtain $a = -1$ and $b = 1$.

b. [4 points] The coordinator also wants a function $g(x) = cx - e^x$, where c is a constant, so that $g(x)$ has at least one critical point. What condition(s) on c will make this true? Find the x-values of all critical points in this case. Your answer may be in terms of c.

Solution: The function $g(x)$ has critical points at values of x that satisfy

$$g'(x) = c - e^x = 0.$$

Then the only potential critical point is $x = \ln(c)$. This critical point exists only if $c > 0$.