2. [8 points] Consider the family of functions

$$g(x) = a\ln(x) + \frac{b}{x},$$

defined for $x > 0$, where a and b are positive constants.

a. [2 points] Any function $g(x)$ in this family has only one critical point. In terms of a and b, what is the x-coordinate of that critical point? Show your work.

Solution: Since

$$g'(x) = \frac{a}{x} - \frac{b}{x^2} = \frac{ax-b}{x^2},$$

we see that $g'(x) = 0$ when $ax - b = 0$, i.e. when $x = \frac{b}{a}$.

Answer: $x =$ b/a

b. [3 points] Is the critical point a local maximum, a local minimum, or neither? Circle your answer below. Use calculus, and be sure to show enough evidence to justify your answer.

Solution: We use the 2nd derivative test. Since

$$g''(x) = -\frac{a}{x^2} + \frac{2b}{x^3} = \frac{-ax+2b}{x^3},$$

we have that

$$g''\left(\frac{b}{a}\right) = \frac{-a\frac{b}{a} + 2b}{\left(\frac{b}{a}\right)^3} = \frac{(-b+2b)a^3}{b^3} = \frac{a^3}{b^2} > 0.$$

Answer: local max **[local min]** neither

c. [3 points] Find values of a and b such that $g(x)$ has a critical point at $(e^2, 1)$. Show your work.

Solution: We know that $0 = g'(e^2) = \frac{ae^2-b}{e^4} = 0$, so that $b = ae^2$. We also know that $1 = g(e^2) = a\ln(e^2) + \frac{b}{e^2} = 2a + \frac{ae^2}{e^2} = 3a$, so that $a = 1/3$. Then $b = e^2/3$.

Answer: $a =$ $1/3$ **Answer:** $b =$ $e^2/3$