9. [8 points] A home improvement store is designing a new bucket to sell. The bucket will be in the shape of a cylinder, so that the volume V of the bucket is given by

$$V = \pi r^2 h,$$

where r is the bucket's radius and h is the bucket's height, both measured in feet. Since the bucket does not have a top, the surface area A of the bucket is given by

$$A = 2\pi r h + \pi r^2.$$

The store has decided that the new bucket should have a volume of exactly 1 cubic foot. Find the dimensions of the bucket that will minimize its surface area. Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact minimize the surface area.

Solution: Since we want the bucket to have a volume of 1, we have $1 = \pi r^2 h$, or $h = \frac{1}{\pi r^2}$. Thus

$$A = 2\pi r\left(\frac{1}{\pi r^2}\right) + \pi r^2 = \frac{2}{r} + \pi r^2,$$

and

$$\frac{dA}{dr} = -\frac{2}{r^2} + 2\pi r.$$

Setting this derivative equal to 0, we find that $2\pi r = \frac{2}{r^2}$, so $\pi r^3 = 1$, and $r = \frac{1}{\pi^{1/3}}$.
To see that this is the global minimum, we use the 2nd derivative test:

$$\frac{d^2A}{dr^2} = \frac{4}{r^3} + 2\pi > 0,$$

so $r = \frac{1}{\pi^{1/3}}$ is a local minimum. Since this was the only critical point on the domain of $(0, \infty)$, it must be the global minimum. Finally, when $r = \frac{1}{\pi^{1/3}} = \pi^{-1/3}$, we see that $h = \frac{1}{\pi\pi^{-2/3}} = \frac{1}{\pi^{1/3}}$.

Answer: surface area is minimized when $r = \frac{1}{\pi^{1/3}}$ $h = \frac{1}{\pi^{1/3}}$