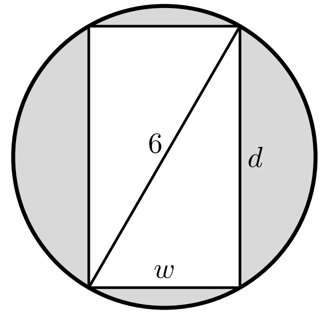
8. [9 points]

A rectangular wooden beam is to be cut from circular tree log of diameter 6 inches, with the rectangular cross section shown in the figure to the right. The strength S of the beam is proportional to the product of the beam's width w in inches and the square of its depth d in inches, so

$$S = kwd^2$$

where $k > 0$ is a constant. Find the dimensions of the beam of maximum strength that can be cut from the log.



Solution: We want to maximize the strength $S = kwd^2$ of the beam, so we first need to write S as a function of one variable. By the Pythagorean Theorem, we have $w^2 + d^2 = 6^2 = 36$, so $d^2 = 36 - w^2$, and therefore

$$S = kw(36 - w^2) = 36kw - kw^3.$$

Since $0 < w < 6$, we seek to maximize $S = S(w) = 36kw - kw^3$ on the interval $0 < w < 6$. Differentiating with respect to w (and recalling that k is a positive constant), we get

$$\frac{dS}{dw} = 36k - 3kw^2 = 3k(12 - w^2).$$

Thus $\frac{dS}{dw} = 0$ when $w = \pm\sqrt{12}$, so the only critical point of $S(w)$ in the interval $(0, 6)$ is $w = \sqrt{12}$. To confirm that $w = \sqrt{12}$ is a maximum of $S(w)$ on $(0, 6)$, note that $S(\sqrt{12}) = 24k\sqrt{12} > 0$ while

$$\lim_{w \to 0^+} S(w) = S(0) = 0 \qquad \text{and} \qquad \lim_{w \to 6^-} S(w) = S(6) = 36k \cdot 6 - k \cdot 6^3 = 0.$$

Finally, using the constraint equation $w^2 + d^2 = 36$, we see that $d = \sqrt{24}$ when $w = \sqrt{12}$.

Answer: $w =$ **$\sqrt{12}$ inches** and $d =$ **$\sqrt{24}$ inches**