8. [8 points] After its first month of production, Alana's company has manufactured 1.2 million Stan Lee cups. Now she is trying to decide whether it would be in the company's interest to halt production here, or else continue manufacturing more cups.

Suppose $C(q)$ and $R(q)$ are the cost and revenue functions, respectively, of the company producing and selling q million cups, and let $\pi(q) = R(q) - C(q)$ be the profit function. Assume $C(q)$ and $R(q)$ are differentiable for all $q > 0$, and that Alana can sell every cup she produces.

The parts below describe different scenarios that are independent of each other. In each scenario, determine whether Alana should *continue producing more cups* or instead *halt production at 1.2 million cups* in order to maximize profit. If there is not enough information to decide, answer NEI. Circle the one best answer.

a. [1 point] Suppose $MR(1.2) > MC(1.2)$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION NEI

b. [1 point] Suppose $C(1.2) > R(1.2)$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION NEI

c. [2 points] Suppose $\pi'(1.2) = 0$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION NEI

d. [2 points] Suppose $\pi'(q) < 0$ for all $q > 1.2$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION NEI

e. [2 points] Suppose $\pi(q)$ has a local minimum at $q = 1.2$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION NEI

9. [11 points] Consider the family of ellipses $\mathcal{C}$ defined implicitly by the equation

$$k^{-4}x^2 + e^{2k}y^2 = 1,$$

where $k > 0$ is a parameter.

a. [2 points] Find the unique value of k such that the point $(0, \frac{1}{2})$ lies on the ellipse $\mathcal{C}$.

Solution: We plug $x = 0$ and $y = \frac{1}{2}$ into the equation that defines $\mathcal{C}$, and solve for k:

$$\begin{aligned} k^{-4} \cdot 0^2 + e^{2k}(1/2)^2 &= 1, \\ e^{2k} &= 4, \\ k &= \frac{1}{2}\ln 4 = \ln 2. \end{aligned}$$

Answer: $k =$ $\ln 2$

This problem continues on the next page

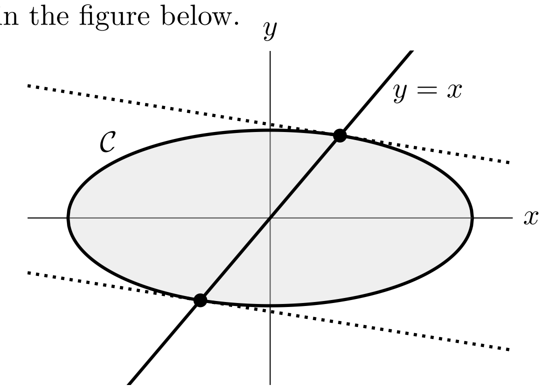
Recall from the previous page that $\mathcal{C}$ is the family of ellipses defined implicitly by the equation

$$k^{-4}x^2 + e^{2k}y^2 = 1,$$

where $k > 0$ is a parameter. The curve $\mathcal{C}$ is pictured for $k = 1$ in the figure below.

b. [4 points]

For every $k > 0$, the ellipse $\mathcal{C}$ intersects the line $y = x$ in two points, as shown in the figure to the right for $k = 1$. The lines tangent to $\mathcal{C}$ at these two points are parallel to each other. Find the *slope* of these dotted lines, in terms of the parameter k.



Solution: The slope of the dotted lines will be the value of $\frac{dy}{dx}$ at the two points given in the picture. So we implicitly differentiate the equation that defines $\mathcal{C}$ with respect to x:

$$2k^{-4}x + 2e^{2k}y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = \frac{-2k^{-4}x}{2e^{2k}y} = \frac{-1}{k^4e^{2k}} \cdot \frac{x}{y}.$$

Since $x = y$ at the two points in question, the term $\frac{x}{y}$ in our expression above reduces to 1, and we obtain a slope of $\frac{dy}{dx} = \frac{-1}{k^4e^{2k}} = -k^{-4}e^{-2k}$.

Answer: slope = $-k^{-4}e^{-2k}$

c. [5 points] The area A of the elliptical region bounded by $\mathcal{C}$ is given in terms of k by

$$A = \pi k^2 e^{-k}.$$

Find the value of k (with $k > 0$) that makes the area of this region as large as possible. Use calculus to find your answer, and show enough evidence to justify your conclusions.

Solution: We need to maximize the function $A = f(k) = \pi k^2 e^{-k}$ on the domain $k > 0$. Taking a derivative, we get

$$\frac{dA}{dk} = 2\pi k e^{-k} - \pi k^2 e^{-k} = \pi k e^{-k}(2 - k).$$

This gives us critical points of $k = 0$ (which we may ignore, since it lies outside our domain) and $k = 2$. Note that $f(2) = \frac{4\pi}{e^2} > 0$. Considering the endpoints of our domain, we have

$$\lim_{k\to 0^+} \frac{dA}{dk} = 0 = \lim_{k\to\infty} \frac{dA}{dk},$$

so $k = 2$ must be the global maximum of $f(k)$ on the interval $(0, \infty)$.

Answer: $k =$ 2