10. [10 points] Let $m(x)$ be a twice-differentiable function that is defined for all real numbers. Suppose the *only* critical point of $m(x)$ is $x = 0$, and that

$$m''(x) = \frac{x^2(9 - x^4)}{(x^4 + 2)^3}.$$

a. [4 points] Find the intervals of concavity of $m(x)$. That is, find the largest open intervals on which $m(x)$ is concave up, and the largest open intervals on which $m(x)$ is concave down. *Show enough work to fully justify your conclusions.*

Solution: We know that $m(x)$ is concave up on intervals where $m''(x) > 0$ and concave down on intervals where $m''(x) < 0$. From the given formula for $m''(x)$, we see that $m''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, so we need to find the sign of $m''(x)$ on the intervals determined by these three points. Notice that $(x^4 + 2)^3$ and x^2 are always nonnegative, so the sign of $m''(x)$ will be the same as the sign of $9 - x^4 = (3 + x^2)(3 - x^2)$. The term $3 + x^2$ is always positive, while $3 - x^2$ is postive only when $-\sqrt{3} < x < \sqrt{3}$. Thus we have the following signs for $m''(x)$:

$m''(x)$:	$-$	$-\sqrt{3}$	$+$	0	$+$	$\sqrt{3}$	$-$

This means $m(x)$ is concave down on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$, and concave up on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$. In fact, since $m''(x) \geq 0$ for all $\sqrt{3} < x < \sqrt{3}$, we see that $m'(x)$ is increasing on the entire interval $(-\sqrt{3}, \sqrt{3})$, so $m(x)$ is concave up on the entire interval $(-\sqrt{3}, \sqrt{3})$.

Answer: Intervals on which $m(x)$ is **concave up**: $(-\sqrt{3}, \sqrt{3})$

Answer: Intervals on which $m(x)$ is **concave down**: $(-\infty, -\sqrt{3})$ **and** $(\sqrt{3}, \infty)$

b. [1 point] Using your work in part (a), list all inflection points of $m(x)$, separated by commas. *No additional justification necessary.*

Answer: $m(x)$ has inflection points at $x =$ $-\sqrt{3}$ **and** $\sqrt{3}$

c. [3 points] Using your work above, classify $x = 0$ as a LOCAL MAX of $m(x)$, a LOCAL MIN of $m(x)$, or NEITHER by circling your answer below, or else circle NEI if there is not enough information to tell. *Include a brief justification of your answer.*

LOCAL MAX [✓ LOCAL MIN] NEITHER NEI

Solution: Since $m(x)$ is differentiable everywhere and has a critical point at $x = 0$, we know $m'(0) = 0$, so $m(x)$ has a horizontal tangent line at $x = 0$. Since we also know $m(x)$ is concave up on $(-\sqrt{3}, \sqrt{3})$, it follows that $m(x)$ has a LOCAL MIN at $x = 0$.

d. [2 points] Find the following limits. Write DNE for any limit that does not exist, even if the limit tends to $\pm\infty$.

i. $\lim\limits_{x \to \infty} m''(x)$

Answer: 0

ii. $\lim\limits_{x \to 0} \dfrac{m''(x)}{x^2}$

Answer: $9/8$