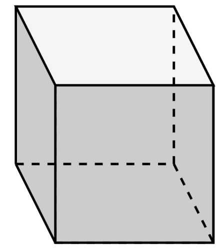
7. [9 points] Suppose 16 square feet of material is available to make a box with a square base and an open top. Find the side length of the base that maximizes the volume of the box.

Show all your work, include units, and fully justify using calculus that you have in fact found side length that maximizes volume.



Solution: Let us write x for the side length of the base, and h for the height of the box. Then the volume and surface area of the box are

$$V = x^2 h \qquad \text{and} \qquad A = x^2 + 4xh,$$

respectively. So we want to maximize V subject to the constraint that $A = 16$. To do this, we need to write V as a function of one variable, so we use the constraint equation to solve for h in terms of x:

$$16 = x^2 + 4xh, \qquad \text{so} \qquad 16 - x^2 = 4xh, \qquad \text{so} \qquad \frac{16 - x^2}{4x} = h.$$

Plugging $h = \frac{16-x^2}{4x}$ into our expression for V, we get

$$V \;=\; V(x) \;=\; x^2\left(\frac{16 - x^2}{4x}\right) \;=\; 4x - \frac{x^3}{4}.$$

We want to maximize $V(x)$ over the interval $0 < x < 4$, since $x \leq 0$ and $x \geq 4$ would not make sense in the context of the problem. Differentiating to find critical points, we have

$$V'(x) \;=\; 4 - \frac{3}{4}x^2.$$

Setting this equal to zero and solving, we find that $V(x)$ has critical points at $\pm\sqrt{16/3}$. The negative value is outside our domain, so the only critical point in our domain is $x = \sqrt{16/3}$. To check that this is indeed a maximum, we could apply the First or Second Derivative Test, or we could check the behavior of $V(x)$ at the endpoints $x = 0$ and $x = 4$. Plugging 0 and 4 into V, we find that $V(0) = V(4) = 0 < V\left(\sqrt{16/3}\right) = \frac{8}{3}\sqrt{16/3}$, confirming that $x = \sqrt{16/3}$ really is the side length, in feet, that maximizes the volume of the box.

Answer: side length of base = $\sqrt{16/3}$ **feet**