10. [11 points] A function $P = f(r)$ and its first and second derivatives are given by the formulas

$$f(r) = \frac{c^2 r}{(r+k)^2} \qquad f'(r) = \frac{c^2(k-r)}{(r+k)^3}, \qquad \text{and} \qquad f''(r) = \frac{2c^2(r-2k)}{(r+k)^4},$$

where c and k are **positive** constants. Note that $f(r)$ is defined for all real numbers *except* $r = -k$. In this problem, you must use calculus to find and **justify** your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [4 points] Find all **local extrema** of $f(r)$, and classify each as a max or a min. If there are none of a particular type, write NONE. Your answer may involve c and/or k.

Solution: Since $f'(r)$ exists everywhere but $r = -k$, and $f(r)$ is undefined at $r = -k$, any local extrema of $f(r)$ must occur at points where $f'(r) = 0$. But $f'(r) = 0$ only when $r = k$. To check whether $r = k$ is indeed a local extremum of $f(r)$, and to determine which type if it is, we use the First Derivative Test.

For $0 < r < k$, we have $c^2 > 0$, $k - r > 0$, and $(r+k)^3 > 0$, so $f'(r) > 0$. For $k < r$ we have $c^2 > 0$, $k - r < 0$, and $(r+k)^3 > 0$, so $f'(r) < 0$. Thus $f'(r)$ changes sign from positive to negative at $r = k$, making k a local maximum of $f(r)$ by the First Derivative Test.

Note: $f'(r)$ changes sign from negative to positive at $r = -k$, but $-k$ is not a local minimum of $f(r)$ since $f(r)$ is undefined at $r = -k$. In fact, $f(r)$ has a vertical asymptote at $r = -k$, with $\lim_{r\to -k} f(r) = -\infty$.

Answer: Local min(s) at $r =$ **NONE** and Local max(es) at $r =$ k

b. [4 points] Find all **global extrema** of $f(r)$ **on the interval** $[0, \infty)$. If there are none of a particular type, write NONE. Your answer may involve c and/or k.

Solution: The function $f(r)$ is continuous on $[0,\infty)$, so we just need to check the values of $f(r)$ at all its critical points in $[0,\infty)$, as well as the behavior of $f(r)$ at the endpoints of $[0,\infty)$. In part **a**., we found $r = k$ to be the only critical point of $f(r)$. Since

$$f(0) = 0, \qquad f(k) = \frac{c^2 k}{(2k)^2} > 0, \qquad \text{and} \qquad \lim_{r\to\infty} f(r) = 0,$$

we find that $f(r)$ has a global min of 0 on $[0,\infty)$ at the point $r = 0$, and a global max of $\frac{c^2 k}{4k^2} = \frac{c^2}{4k}$ on $[0,\infty)$ at the point $r = k$.

Answer: Global min(s) at $r =$ 0 and Global max(es) at $r =$ k

c. [3 points] Find values of the positive constants c and k so that the function $P = f(r)$ has an inflection point at $r = 4$ and a global maximum value of $P = 8$. *Show all work.*

Solution: Since $f''(4)$ exists, in order for $f(r)$ to have an inflection point at $r = 4$ we must have $f''(4) = 0$. This means $4 - 2k = 0$, so $k = 2$. In part **c**., we found $f(r)$ has a global maximum value of $\frac{c^2}{4k}$, so by setting $\frac{c^2}{4k}$ equal to 8 and plugging in $k = 2$, we find $c = 8$.

Answer: $c =$ 8 and $k =$ 2