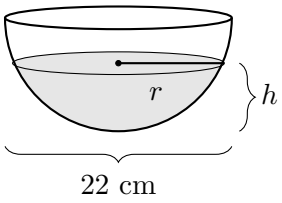
12. [9 points]

A hemispherical bowl of radius 11 cm is being filled with water at the constant rate of π cm^3/sec. When the depth of the water from the bottom of the bowl is h centimeters, the volume of the water in the bowl in cubic centimeters is given by

$$V \;=\; 11\pi h^2 - \frac{\pi}{3}h^3.$$



a. [4 points] At what rate is the depth of water in the bowl increasing when the depth of the water is 2 centimeters, in units of centimeters per second?

Solution: We are given that $\frac{dV}{dt} = \pi$, and we are asked to find $\frac{dh}{dt}$ when $h = 2$. Differentiating $V = 11\pi h^2 - \frac{\pi}{3}h^3$ with respect to t gives us

$$\frac{dV}{dt} \;=\; 22\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt} \;=\; \pi h(22-h)\frac{dh}{dt}.$$

Plugging in $h = 2$ and using $\frac{dV}{dt} = \pi$, we get

$$\pi \;=\; \frac{dV}{dt} \;=\; \pi(2)(22-2)\frac{dh}{dt} \;=\; 40\pi\frac{dh}{dt}, \qquad \text{so} \qquad \frac{dh}{dt} \;=\; \frac{\pi}{40\pi} \;=\; \frac{1}{40}.$$

Answer: 1/40 cm/sec

b. [5 points] As the bowl fills up, the circular surface of the water grows larger. At what rate is the area of the water's surface increasing when the depth of the water is 2 centimeters (as in part **a**)? *Include units.*

Solution: Let r be the radius, and A the area, of the circular surface of the water. We must find $\frac{dA}{dt}$ when $h = 2$. Differentiating both sides of $A = \pi r^2$ with respect to t, we get

$$\frac{dA}{dt} \;=\; 2\pi r\frac{dr}{dt}. \tag{1}$$

To find $\frac{dr}{dt}$, we need an equation relating r and h. From the picture, we see that the line segment labeled "r" and the segment of length $11-h$ from the center of the circular surface of the water to the center of the hemispherical bowl form two legs of a right triangle with hypotenuse 11 cm. Therefore by the Pythagorean Theorem we have

$$r^2 + (11-h)^2 = 11^2, \qquad \text{so} \qquad r^2 = 22h - h^2.$$

Differentiating this gives us

$$2r\frac{dr}{dt} \;=\; 22\frac{dh}{dt} - 2h\frac{dh}{dt}. \tag{2}$$

Now we can solve for $\frac{dA}{dt}$ when $h = 2$ by combining equations (1) and (2) and plugging in $h = 2$ and $\frac{dh}{dt} = \frac{1}{40}$ from part **a**:

$$\begin{aligned}\frac{dA}{dt} \;=\; 2\pi r\frac{dr}{dt} \;&=\; \pi\left(22\frac{dh}{dt} - 2h\frac{dh}{dt}\right)\\ &=\; \pi\left(22\frac{1}{40} - 2(2)\frac{1}{40}\right) \;=\; \frac{18\pi}{40} \qquad \text{when } h = 2.\end{aligned}$$

Answer: $\frac{18\pi}{40}$ **cm^2/sec**