MATH 116 — FINAL EXAM

December 15, 2003

NAME: ______________________________

INSTRUCTOR: ____________________ SECTION NO: ____________________

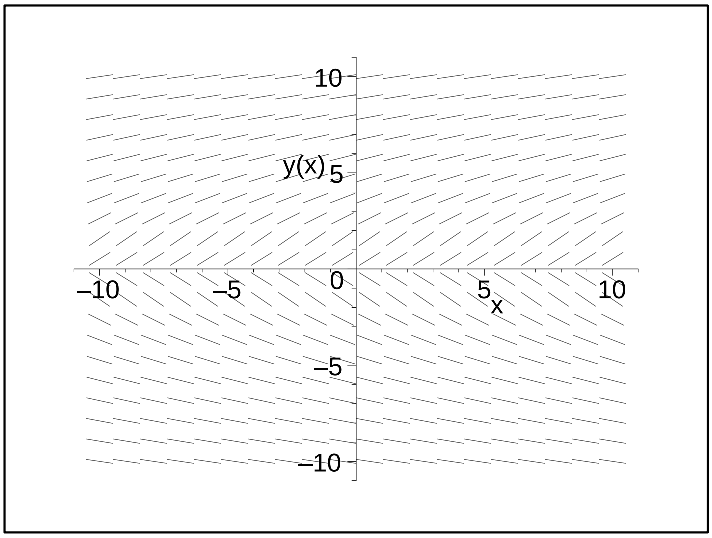
1. Do not open this exam until you are told to begin.
2. This exam has 12 pages including this cover. There are 10 problems.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	6	
2	8	
3	10	
4	8	
5	12	
6	14	
7	10	
8	12	
9	15	
10	5	
TOTAL	100	

1. (6 points) Which of the following differential equations has the slope field given in the figure? (Circle the letter of each correct answer.)

a. $\frac{dy}{dx} = \frac{2x}{1+x^2}$ b. $\frac{dy}{dx} = e^{-y^2}$ c. $\frac{dy}{dx} = \frac{2x^2}{1+x^4}$

d. $\frac{dy}{dx} = \frac{2y}{1+y^2}$ e. $\frac{dy}{dx} = e^{-x^2}$ f. $\frac{dy}{dx} = \frac{2y^2}{1+y^4}$



2. (8 points) Circle "True" or "False" for each of the following statements. No explanation is necessary. (Remember that "True" means the statement is always true.)

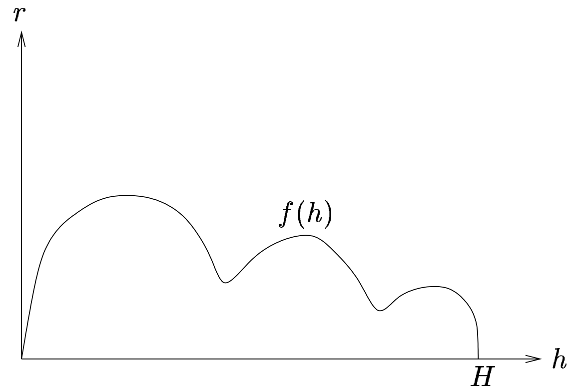
(a) The function $y(t) = 0$ is an equilibrium solution of the differential equation $dy/dt = y + t$.

True. False.

(b) If $P(t)$ is a solution of the logistic differential equation, $dP/dt = .5P(200 - P)$, then so is the function $2P(t)$.

True. False.

3. (10 points) The cross-sections of a snowman are given by circles of radius $r = f(h)$ where h is the height measured from the ground and $f(h)$ has graph given in the figure. Both r and h are measured in inches.



(a) Draw and label a typical, thin, cross-section of the snowman. What is the volume of the cross-section (in terms of the function $f(h)$)?

(b) Write an integral in terms of $f(h)$ whose value is the total amount of snow used in making the snowman.

4. (8 points) **(a)** Give the Taylor series about the point t=0 of the function

$$f(t) = \frac{\sin t}{t}.$$

(You are allowed to use the standard Taylor series expansions without deriving them).

(b) Is the following statement "True" or "False"? Explain why, if true, or why not, if false.

The Taylor series about the point $x = 0$ of the sine integral *function, defined by*

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt,$$

is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots.$$

True. False.

5. (12 points) **(a)** Carry out two steps of Euler's method with $\Delta x = .1$ to approximate the solution $y(x)$ of the initial value problem,

$$\frac{dy}{dx} = xy + y^3, \qquad y(2) = 1.$$

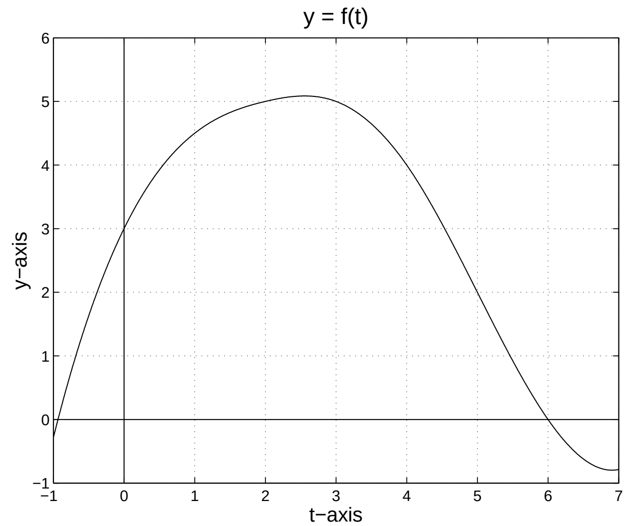
at $x = 2.2$. To receive credit, you must write out the calculations that are needed to make the computation. (No credit for answers alone, even correct ones.)

(b) Will the solution be an increasing function of x for $x \geq 2$? Explain.

(c) Will your estimate of $y(2.2)$ be an underestimate or an overestimate? Explain.

6. (14 points) The function f is defined for $-1 \le t \le 7$ and has graph given in the figure below. The function F is defined by

$$F(x) = \int_2^x f(t)\, dt.$$



(a) Fill in the following table of values of $F(x)$ and $F'(x)$, using the best approximation to the values of these functions that you can determine using the given graph of f.

x	0	2	4	6
$F(x)$				
$F'(x)$				

(b). Compute $g'(2)$, where g is the function defined by $g(x) = F(x^2)$. (Show your work.)

(c) On which subintervals (approximately), if any, of $-1 \le t \le 7$ is F concave up?

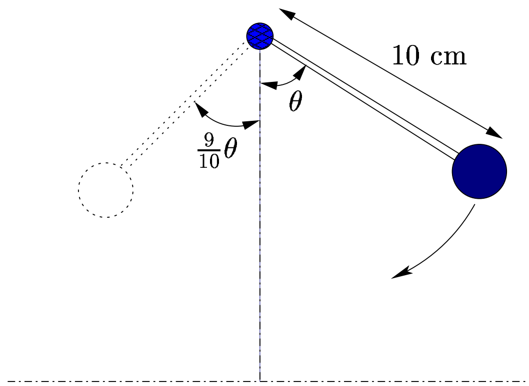
7. (10 points) During the holiday season there are two main groups of people at the mall (excluding the store employees). These are the shoppers and the volunteers ringing bells to collect money for charities. The numbers of each vary over time. If we let $B(t)$ be the number of bell ringers at time t and $S(t)$ be the number of shoppers at time t, and assume these are modeled by a predator-prey system of differential equations, then the differential equations describing their numbers are

$$\begin{aligned} \frac{dB}{dt} &= -1,000\,B + 2\,BS \\ \frac{dS}{dt} &= 66\,S - 11\,BS. \end{aligned}$$

(a) Given this model, which is the "predator" and which is the "prey"? Make sure you justify your answer by explaining how this is reflected in the given equations.

(b) What are the equilibrium points of this system? Describe what the equilibrium points mean in terms of this problem.

8. (12 points) You begin a pendulum swinging in the position shown in the figure below with $\theta = \pi/4$. Assume the pendulum travels in a circular arc, swinging to the left past the center line shown in the figure and then returning to the right. Notice that the pendulum must briefly stop its motion before it can change direction. We define one "swing" of the pendulum to be the motion between the times when the pendulum stops its motion to change direction. For example, the first "swing" is the motion from the time you release the pendulum until it swings all the way to the left. The second "swing" is the motion coming back from the left to the right, and so on.



(a) Assume that at the end of each swing the pendulum makes an angle of $\frac{9}{10}$ the angle it made when it began the swing. What angle does the pendulum make after its second swing? After its third swing? After its n^{th} swing?

(b) Recall that the arc length of a circle is given by the formula $s = r\alpha$ where s is arc length, r is the radius of the circle, and α is the angle measuring the arc length. How far does the weight travel on its first swing? On its second swing? On its n^{th} swing?

Problem continued on next page.

(c) What is the total distance the weight has travelled after 30 swings?

(d) If the pendulum were allowed to swing forever how far would it travel?

9. (15 points) Frodo Baggins of the Shire is given the task of taking the ring of power from the elven kingdom of Rivendell to Mount Doom, 100 km away, to destroy it. The ring's weight w (in kg) grows at a rate of one one-hundredth (1/100) of Frodo's distance x (in km) from Rivendell as Frodo proceeds on his journey. Frodo can travel toward Mount Doom at the rate of 2.5 km per hour except that the weight of the ring of power slows his rate of travel (in km/hr) to Mount Doom by one-twentieth (1/20) of the weight of the ring (in kg). Suppose that the ring weighs .001 kg (1 gram) when he begins his journey from Rivendell.

(a) Write a pair of differential equations for the functions $w(t)$ and $x(t)$ that give the weight of the ring and Frodo's distance from Rivendell on the road to Mount Doom t hours after leaving Rivendell. What are the initial conditions at time $t = 0$?

(b) What differential equation models the relationship between the weight of the ring and the distance from Rivendell?

Problem continued on next page.

(c) Find the trajectory that Frodo follows in the x-w phase plane by solving the differential equation you found in part (b). (Show your work. Don't forget the initial conditions from part (a)).

(d) Will Frodo complete his task by traversing the 100 km. from Rivendell to Mount Doom or will he fall short of accomplishing this goal? Explain why or why not. (Hint: Frodo is stopped if $\frac{dx}{dt} = 0$, i.e., the weight of the ring becomes too much for him.)

10. (5 points) Suppose that on your visit home over break you meet a friend who is now taking precalculus at your old high school. He knows the formula "distance travelled = rate $\times$ time." He also knows some students who are taking the calculus course at the high school, and has heard from them there is a more general formula, "distance travelled = area under the velocity curve," that computes the distance, even when the velocity is not constant. He asked those students to explain this second formula, but they just shrugged and said he would have to wait until he learned calculus to get an explanation.

Write down what you would tell your friend to explain why the second formula holds and how it is related to the formula he has learned in precalculus. Be sure to include any appropriately labelled graphs you might draw in making your explanation.

Please **print** your name here: Name ______________________________