MATH 116 — FIRST MIDTERM EXAM

Fall 2003

NAME: ______________________________ ID NUMBER: ____________________________

INSTRUCTOR: ______________________ SECTION NO: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	7	
2	10	
3	6	
4	10	
5	10	
6	10	
7	10	
8	10	
9	12	
10	15	
TOTAL	100	

1. (7 points) The *sine-integral* function $Si(x)$ is defined by

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

What is the derivative of $Si(x^3)$?
Answer: We use the 2nd Fundamental Theorem and and the chain rule to arrive at our answer. According to that theorem, the derivative of $Si(x)$ is $Si'(x) = \sin(x)/x$. Therefore, by the chain rule,

$$\begin{aligned} \frac{d}{dx} Si\left(x^3\right) = Si'(x^3)\frac{d}{dx}x^3 &= \frac{\sin(x^3)}{x^3} \times 3x^2 \\ &= \frac{3\sin(x^3)}{x} \end{aligned}$$

2. (10 points) Let $g(x)$ be a continuously differentiable functions of x that satisfies $g(1) = 2$, $g(5) = 6$, and $\int_1^5 g(x)\, dx = -2$. Compute, showing all your work,
Answers:
(a) We use integration by parts to compute this integral:

$u = x$	$dv = g'(x)dx$
$du = dx$	$v = g(x)$

$$\begin{aligned} \int_1^5 xg'(x)dx &= xg(x) \mid_1^5 - \int_1^5 g(x)dx \\ &= (5g(5) - 1g(1)) - (-2) \\ &= 30. \end{aligned}$$

(b) We use a u-substition for this integral. Let $u = 4x - 7$, so $du = 4dx$.

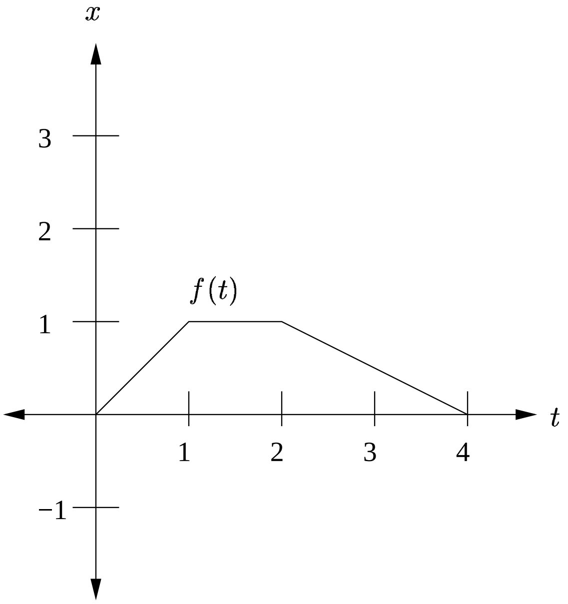
x	u
2	1
3	5

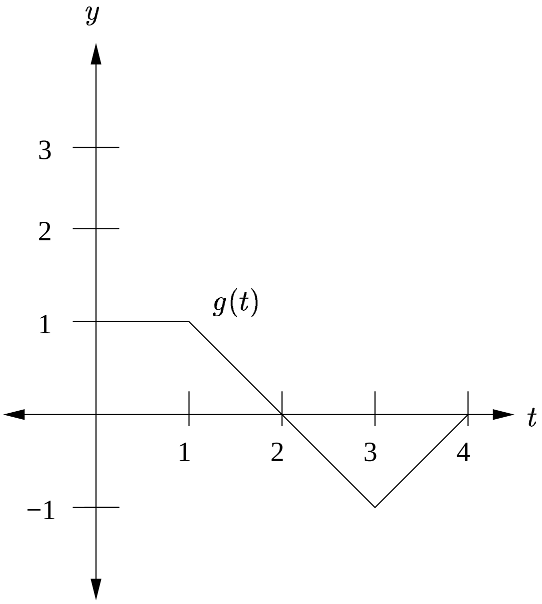
$$\begin{aligned} \int_2^3 g(4x-7)dx &= \frac{1}{4}\int_1^5 g(u)du \\ &= -\frac{2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

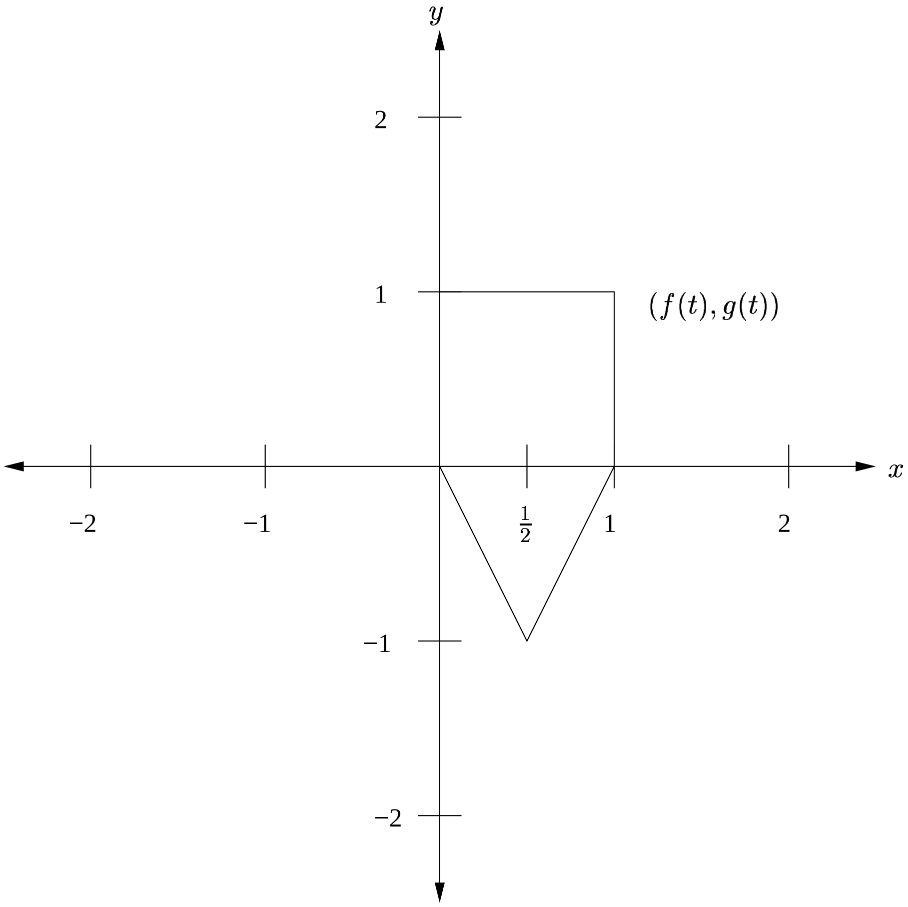
3. (6 points) Let $r(t)$ represent the rate that the height of a child changes per year (in inches per year), where $t = 0$ corresponds to the birth date of the child. Explain the meaning of the quantity $\int_4^8 r(t)\, dt$. (Remember to use units.)
Answer:
The quantity $\int_4^8 r(t)dt$ represents the number of inches a child grows between 4 years of age and 8 years of age.

4. (10 points) Let f, g be the functions with graphs as shown in the first two of the figures below. On the axes given below the graphs of f and g, sketch a graph of the curve with parametric equations $x = f(t)$, $y = g(t)$, $0 \le t \le 4$.

Answer:







5. (10 points) Circle the correct answer(s) to each of the following questions or circle True or False, as appropriate. (No explanations necessary.) In all the questions, f is a continuous function defined on an interval $a \leq x \leq b$.

(a) If f is a decreasing function, then for every $n = 1, 2, 3, \ldots$, the approximation to $\int_a^b f(x)\,dx$ given by LEFT(n) is an

underestimate. [✓ overestimate.] could be either.

(b) If f is concave down, then for every $n = 1, 2, 3, \ldots$, the approximation to $\int_a^b f(x)\,dx$ given by TRAP(n) is an

[✓ underestimate.] overestimate. could be either.

(c) If n is very large, then the midpoint rule MID(n) always gives one the exact value of $\int_a^b f(x)\,dx$.

True. [✓ False.]

(d) The approximation to $\int_a^b f(x)\,dx$ given by the trapezoidal rule TRAP(n) is always more accurate than that given by the left rule, LEFT(n).

True. [✓ False.]

(e) Given a graph of $f'(x)$, one can uniquely determine the graph of $f(x)$.

True. [✓ False.]

6. (10 pts.) A leak is found in the dam your company just finished constructing. The reservoir behind the dam contains 10 million gallons of water when the leak is first discovered, and it is believed that the water is leaking out at a rate of $r(t) = 0.23e^{\frac{t}{1+t}}$ millions of gallons per hour t hours after this time. If it takes your crew 5 hours to repair the leak, how much water would be lost? (Be sure to show your work and explain how arrived at your answer.)

Answer:

The amount of water that has leaked out after 5 hours is equal to the integral from $t = 0$ to $t = 5$ of the rate that water leaks from the dam, or

$$\int_0^5 0.23e^{\frac{t}{t+1}}\,dt.$$

This integral is not readily computable by hand, so one uses a calculator to get an approximate numerical answer. Using the numerical integration function on a TI-83, the result, to two decimal places is

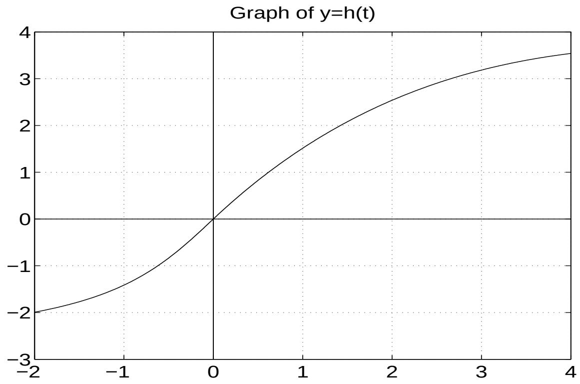
$$\int_0^5 0.23e^{\frac{t}{t+1}}\,dt \simeq 2.22 \text{ millions of gallons.}$$

So after 5 hours, only $10 - 2.22 = 7.78$ millions of gallons remain in the reservoir.

7. (10 points) A function F is defined for $-2 \leq x \leq 4$ by the formula

$$F(x) = \int_0^x e^{-h(t)}\,dt$$

where h is the function with the graph shown below.



Graph of y=h(t)

(a) True or False? For $-2 < x < 4$, $\quad F'(x) = e^{-h(t)}$. T [✓ F]

(b) On which interval or subintervals of $[-2, 4]$ is F increasing? decreasing?

Answer:

Using the second fundamental theorem we see that

$$F'(x) = e^{-h(x)}$$

for all $x \in [-2, 4]$. But we know $e^t > 0$ for all t, so this means that the derivative of $F(x)$ is always positive. Thus $F(x)$ is increasing on the entire interval $[-2, 4]$.

(c) On which interval or subintervals of $[-2, 4]$ is F concave up? concave down?

Answer:

To check concavity, we take the second derivative of $F(x)$:

$$F''(x) = -h'(x)e^{-h(x)}.$$

From the graph we see $h(x)$ is an increasing function on $[-2, 4]$, so $h'(x) > 0$ on $[-2, 4]$. Thus $-h'(x) < 0$ on $[-2, 4]$. Using again that $e^t > 0$ always, we get that $F''(x) < 0$ on $[-2, 4]$ and so $F(x)$ is concave down on the entire interval.

Alternatively, we can observe that $F'(x) = e^{-h(x)}$ is a decreasing function because h and the exponential function are increasing. Therefore, F is concave down.

8. (10 points) Let f be a continuous, positive function for $x \geq 1$.

(a) Define what it means to say that $\int_1^\infty f(x)\,dx$ converges.

Answer:

$\int_1^\infty f(x)dx$ converges if $\lim_{b\to\infty}\int_1^b f(x)dx$ is a finite number.

(b) If f from part (a) is such that $\int_1^\infty f(x)\,dx$ converges and if g is another continuous positive function for $x \geq 1$ that satisfies

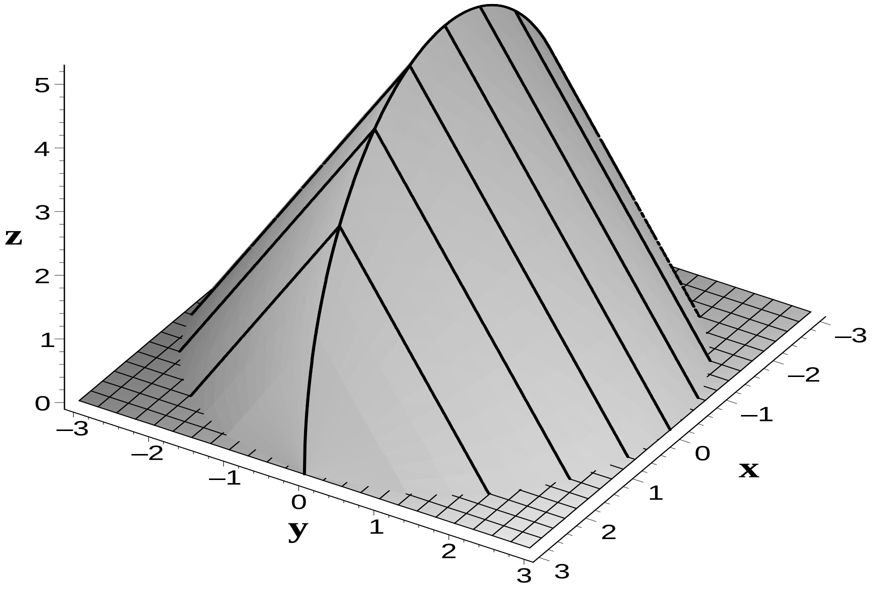
$$g(x) \leq 5f(x) + \frac{3}{x^2}$$

then is it necessarily true that $\int_1^\infty g(x)\,dx$ converges? (Explain why or why not.)

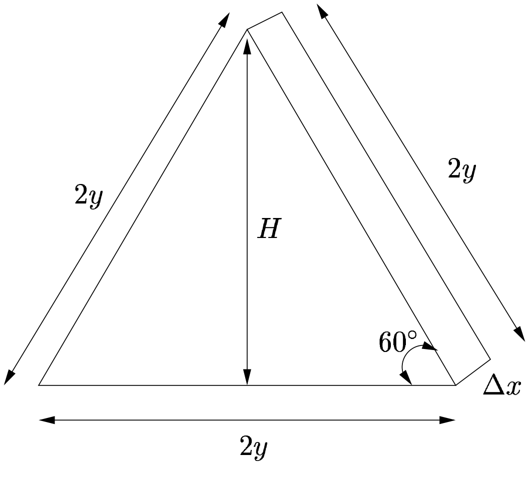
Answer:

The fact that $\int_1^\infty f(x)dx$ converges tells us that $\int_1^\infty 5f(x)dx$ also converges. We also know from class that $\int_1^\infty \frac{1}{x^2}dx$ converges, so $\int_1^\infty \frac{3}{x^2}dx$ converges as well. Therefore, the sum $\int_1^\infty (5f(x)+\frac{3}{x^2})dx$ converges. Since $\int_1^\infty g(x)dx \leq \int_1^\infty (5f(x)+\frac{3}{x^2})dx$, we can conclude by the comparision test that $\int_1^\infty g(x)dx$ converges.

9. (12 points) It's a beautiful sunny day and you are at the beach. You manage to build the most spectacular sand castle ever. Unforunately, fate is cruel and a rogue wave hits the beach and washes over your sandcastle. But, fate also has a kinder side and it leaves you a shapely mound of sand as pictured below. The mound has as a base the interior of the circle $x^2 + y^2 = 9$ in the x-y plane and has cross sections by planes perpendicular to the x-axis given by equilateral triangles with one side in the x-y plane.



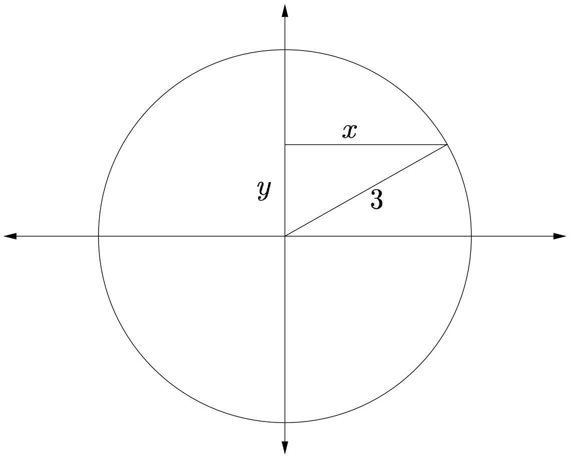
(a) Sketch and label the dimensions of a typical slice of the sand mound perpendicular to the x-axis for $-3 < x < 3$. What is the volume of this slice in terms of x?



The volume of this slice is $V_{\text{slice}} = \frac{1}{2}\,2yH\Delta x = yH\Delta x$. In order to put this in terms of x, we need to express H and y in terms of x. We can use some trigonometry to write $H = 2y\sin 60 = \sqrt{3}y$. To write y in terms of x, we use the fact that we know the base satisfies the equation $x^2 + y^2 = 9$.

From this figure we see that $y = \sqrt{9 - x^2}$. So our formula for the volume of a slice becomes

$$\begin{aligned} V_{\text{slice}} &= 2y\,H\,\Delta x \\ &= \sqrt{3}(\sqrt{9 - x^2})\,(\sqrt{9 - x^2})\,\Delta x \\ &= \sqrt{3}\,(9 - x^2)\Delta x \end{aligned}$$



(b) Write a Riemann sum and then a definite integral representing the volume of the sand pile.

Answer:

The volume of the sand pile can be approximated by adding up all the slices of volume found in part (a). This gives the Riemann sum :

$$\begin{aligned} V_{\text{sand pile}} &= \sum V_{\text{slice}} \\ &= \sum \sqrt{3}\,(9 - x^2)\,\Delta x \end{aligned}$$

Now let $\Delta x \to 0$, so the Riemann sum becomes a definite integral. The volume of the slice is then given by

$$V_{\text{sand pile}} = \sqrt{3}\int_{-3}^{3}(9 - x^2)\,dx$$

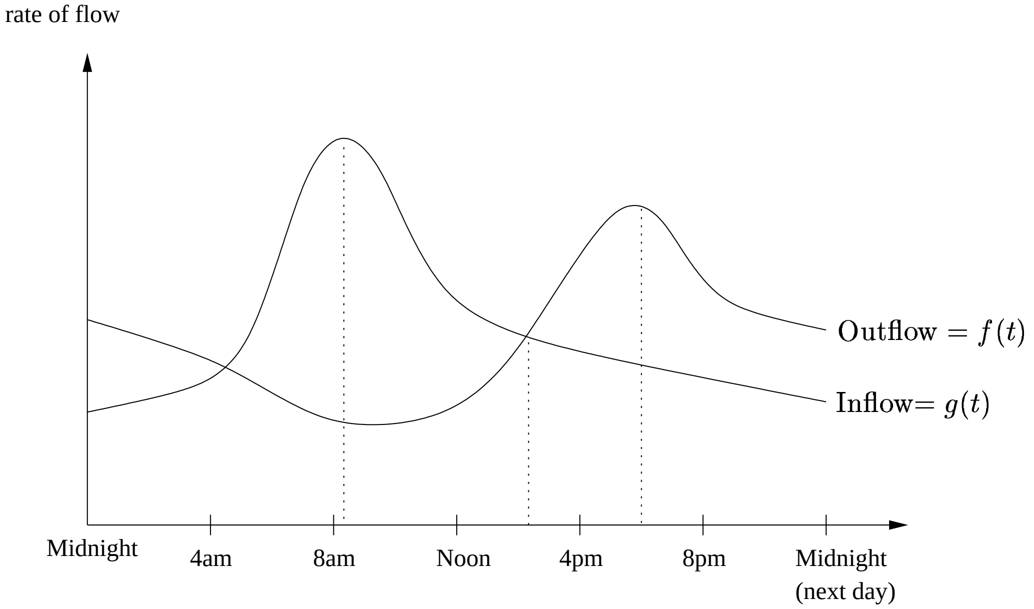
(c) Find the exact volume of the solid. If you can't compute the volume exactly, give the most accurate approximation you can and explain how you found it.

Answer:

This integral is an elementary integral to evaluate, involving only power functions.

$$\begin{aligned} V_{\text{sand pile}} &= \sqrt{3}\int_{-3}^{3}(9 - x^2)\,dx \\ &= \sqrt{3}\left(9x - \frac{1}{3}x^3\right)|_{-3}^{3} \\ &= 36\sqrt{3} \end{aligned}$$

10. (15 points) The graphs shown represent the flow of traffic (in number of motor vehicles per minute) in and out of Ann Arbor on a typical weekday.



(a) During the course of the day, at what time is the largest number of cars in Ann Arbor? Give an explanation of how you arrived at this answer.

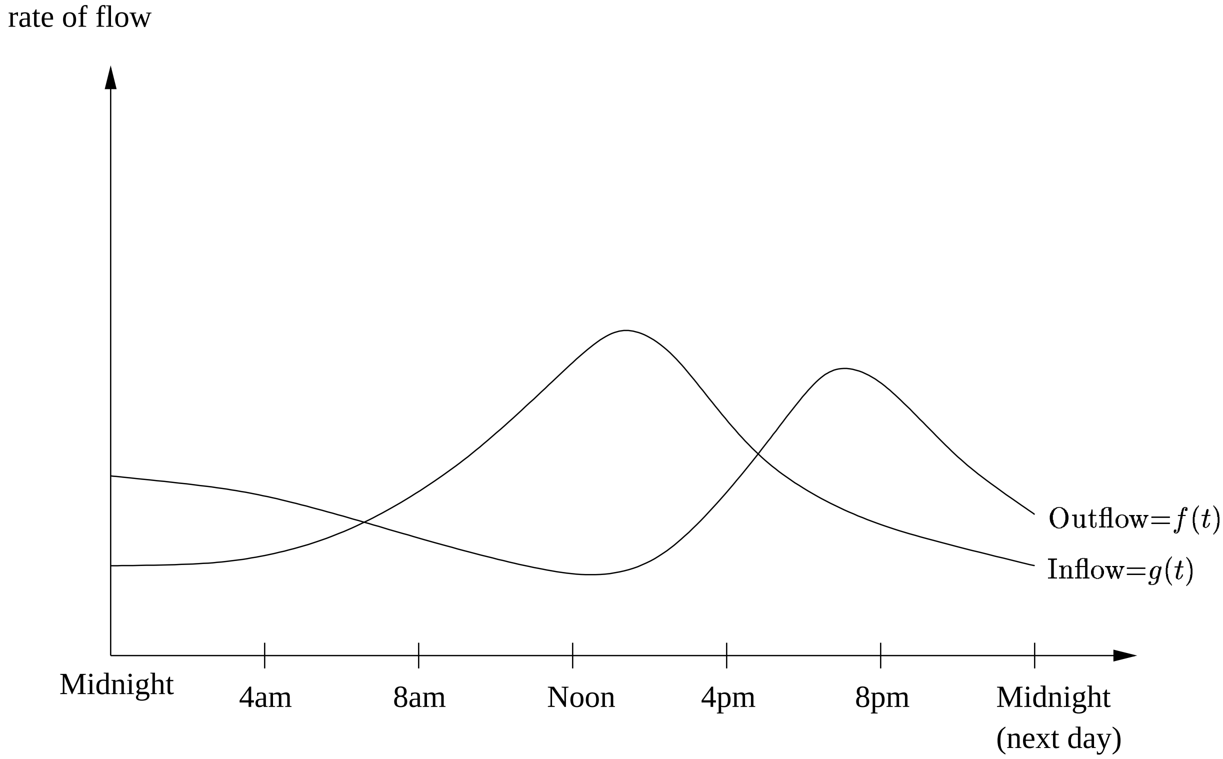
Answer:
The number of cars at time T is equal to the initial number of cars plus $\int_0^T (g(t) - f(t))dt$. In order to find out when the largest number of cars are in Ann Arbor, we want to find when this integral is the largest. This happens around 2:30 pm. The number of cars decreases slightly between midnight and 4:15am, but the amount of increase between 4:15 am and 2:30 more then makes up for this decrease. After 2:30 the integrand is negative, so the number of cars begins to decrease.

(b) At what time is the number of cars in Ann Arbor increasing the most rapidly? Decreasing the most rapidly? Again, please give an explanation of how you arrived at this answer.

Answer:
The number of cars is increasing most rapidly when the rate in - rate out is the largest. This happens at about 8:15 am.
The number of cars is decreasing most rapidly when the rate out - rate in is the largest. This happens at about 6:00 pm.

(c) Sketch possible graphs of the inflow of traffic and the outflow of traffic in for Ann Arbor on a football Saturday if we assume kickoff is at 3pm. Explain how you arrived at the graph drawn.

Answer:
There are many possible answers here, what we want is something reasonable with a good explanation.



Some people will want to leave town early in the morning to get out of the craziness of a football Saturday in Ann Arbor. Around noon people will start coming into town to tailgate and beat the rush. The maximum rate of cars entering will be around 1:30 pm, giving people some time to park and get to their seats before kickoff. The outflow is very low once people start arriving because the people of Ann Arbor know better then to venture from their homes once football traffic has started. The point where the outflow and inflow cross is the time when the maximum number of cars are in Ann Arbor, as we saw in part (a). This occurs at a little after 4 pm. This allows for people who arrived late to the game as well. One could put this at 3pm or around that time and be fine as well. The outflow will start becoming large at about 7 pm as people leave the game early or as the game ends. The maximum outflow will be smaller then the maximum inflow as people will hang around after the game to go to the bars and visit friends.