MATH 116 — SECOND MIDTERM EXAM

November 11, 2003
Solutions

NAME: ______________________________

INSTRUCTOR: ____________________ SECTION NO: ______________________

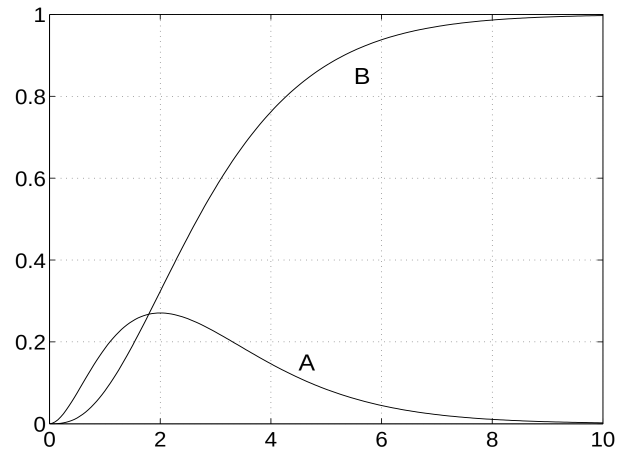
1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	10	
2	8	
3	10	
4	15	
5	5	
6	10	
7	14	
8	12	
9	10	
10	6	
TOTAL	100	

1. (10 points) The figure shows the graphs of two functions, A and B, one of which is a probability density function and the other of which is the corresponding cumulative distribution function.
(a) Which curve represents the density function and which represents the cumulative distribution function?

B is the cumulative density function and A is the probability density function.

(b) Put reasonable values on the tick marks on each of the axes.



2. (8 points) The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n+1}$ is $R =$ $\sqrt{\frac{1}{3}}$.
(Show your work and/or explain your reasoning.)

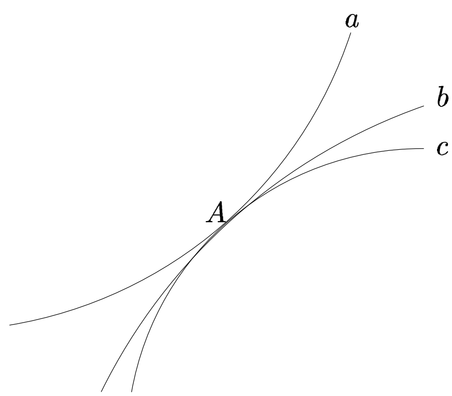
We use the ratio test to find the radius of convergence of this power series.

$$\lim_{n\to\infty} \left| \frac{3^{n+1} x^{2(n+1)}}{n+2} \frac{n+1}{3^n x^{2n}} \right| = \lim_{n\to\infty} \frac{3(n+1)}{n+2} |x^2| = 3|x|^2$$

We know the ratio test yields convergence when the limit is less then 1 and divergence when the limit is greater than 1. So, we have convergence when $3|x|^2 < 1$ or $|x| < 1/\sqrt{3}$ and divergence when $3|x|^2 > 1$ or $|x| > 1/\sqrt{3}$. Therefore, the radius of convergence is $R = 1/\sqrt{3} = \sqrt{\frac{1}{3}}$.

3. (10 points) Three functions f_1, f_2, and f_3, have graphs that pass through a point A and are shown in the figure. Second degree Taylor polynomials for these functions are as follows:

$$\begin{aligned} f_1(x) &\approx 10 + (x-5) - (x-5)^2 \\ f_2(x) &\approx 10 + (x-5) + (x-5)^2 \\ f_3(x) &\approx 10 + (x-5) - 5(x-5)^2 \end{aligned}$$



(a) What are the coordinates of the point A?

$A = (5, 10)$.

(b) Which function goes with which graph? Explain how can you tell?

First we note that all the functions have the same first derivative and value at $x = 5$ by looking at the constant terms and coefficients of the linear terms in the Taylor series. Therefore we will have to decide which function goes with which graph by looking at the second derivative. $f_2(x)$ is the only function with a positive second derivative ($\frac{f^{(2)}(5)}{2!} = 1$), so it must be the function that is concave up. So $f_2(x)$ is given by graph a. The other two functions are both concave down because their second derivatives are negative. However, $f_3(x)$ has a second derivative that is "more negative" then $f_1(x)$, so it will be "more concave down". Therefore, we see that $f_1(x)$ is given by b and $f_3(x)$ is given by c.

4. (15 points) Circle "True" or "False" for each of the following statements. Circle "True" only if the statement is always true. No explanation is necessary.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges.

True. | False. |

(b) If $0 \le a_n \le b_n$ for all n, and if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

| True. | False.

(c) If $P_4(x) = 5 + 6(x-a) + 2(x-a)^2 + 37(x-a)^3 + 21(x-a)^4$ is the 4th degree Taylor polynomial for $f(x)$ about $x = a$, then $f^{(3)}(a) = 37$.

True. | False. |

(d) If the power series $\sum_{n=0}^{\infty} C_n(x-3)^n$ converges for $x = 1$, then it also converges for $x = 4$.

| True. | False.

(e) The infinite series $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^5+3}$ converges.

| True. | False.

5. (5 points) Express the number x whose repeating decimal expansion is 6.17636363636363... as the sum of an infinite series.

$$
\begin{aligned}
6.17636363636\overline{363} &= 6.17 + \frac{63}{10,000} + \frac{63}{1,000,000} + \frac{63}{100,000,000} + \ldots \\
&= 6.17 + 63 \sum_{n=2}^{\infty} \frac{1}{100^n} \\
&= 6.17 + \frac{63}{100} \sum_{n=1}^{\infty} \frac{1}{100^n}.
\end{aligned}
$$

6. (10 points) Einstein's special theory of relativity states that an object's length contracts as its velocity increases according to the formula

$$L(v) = L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

where L_0 is the length of the object at rest, v is the velocity of the object, and c is the speed of light. (Recall from physics that $v < c$ necessarily)

(a) Approximate $L(v)$ by its second degree Taylor polynomial near $v = 0$.

One can use the binomial series expansion given in the text, $\sqrt{1+x} \approx 1 + \frac{1}{2} - \frac{1}{8}x^2 + \ldots$ *or just directly calculate the 2nd degree Taylor polynomial. Using the first method and the linear approximation from the Taylor series expansion of* $\sqrt{1+x}$ *with* $x = -v^2/c^2$ *gives*

$$L(v) \approx L_0\left(1 - \frac{1}{2}\left(\frac{v}{c}\right)^2\right).$$

(b) What is the approximate error in your approximation from part (a) in terms of v when v is small compared to c?

Use the degree two Taylor approximation to $\sqrt{1+x}$ *with* $x = -v^2/c^2$ *to find the Taylor polynomial of degree 4 approximating* $L(v)$ *near* $v = 0$ *is*

$$P_4(v) = L_0\left[1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 - \frac{1}{8}\left(\frac{v}{c}\right)^4\right]$$

so the error $E(v) = L(v) - L_0\left[1 - \frac{1}{2}\left(\frac{v}{c}\right)^2\right]$ *when* $L(v)$ *is approximated by the second degree Taylor polynomial is*

$$E(v) \approx -\frac{L_0}{8}\left(\frac{v}{c}\right)^4.$$

(c) By what percentage will the length of the object contract when it is travelling at a velocity of 99% of the speed of light?

In order to calculate the percentage the length will contract, we need to calculate $L(.99c)$.

$$\begin{aligned} L(0.99c) &= L_0\sqrt{1-\left(\frac{0.99c}{c}\right)^2} \\ &= 0.14L_0 \end{aligned}$$

Therefore, the ship's length has shrunk by approximately 86%. $(1 - .14 = .86)$.

7. (14 points) Suppose the amount of time x that riders wait for the bus to arrive at a certain bus stop is given by the probability density function

$$f(x) = \frac{1}{10}e^{-\frac{1}{10}x}$$

where x is measured in minutes.

(a) What percentage of the time will a rider wait less than 5 minutes for the bus to arrive? (Show your work.)

Knowing the formula to use is the most important part here and should be given most of the credit. It is ok to use their calculators to calculate the integral, and integration errors should not have a large number of points deducted.

The percentage of time the rider will wait less then 5 minutes is given by the probability that the waiting time is less then 5 minutes. This is given by

$$\begin{aligned} \textit{Waiting time} < 5\,\textit{minutes} &= \int_0^5 f(x)dx \\ &= \frac{1}{10}\int_0^5 e^{-\frac{1}{10}x}dx \\ &= \int_0^{\frac{1}{2}} e^{-u}du \\ &= 1 - e^{-\frac{1}{2}} \\ &= 0.39 \end{aligned}$$

So 39% of the time a rider will wait less then 5 minutes for the bus.

(b) What is the mean waiting time until the next bus arrives? (Show your work.)

The mean waiting time is given by

$$\begin{aligned} \textit{mean waiting time} &= \int_0^\infty xf(x)dx \\ &= \frac{1}{10}\int_0^\infty xe^{-\frac{1}{10}x}dx \\ &= \lim_{b\to\infty} -xe^{-\frac{1}{10}x}\Big|_0^b + \frac{1}{10}\int_0^\infty 10e^{-\frac{1}{10}x}dx \quad \textit{(integration by parts)} \\ &= 10\int_0^\infty e^{-u}du \\ &= 10 \quad \textit{minutes.} \end{aligned}$$

(c) What is the median waiting time until the next bus arrives? (Show your work)

The median waiting time T is given by

$$\begin{aligned} 0.5 &= \int_0^T f(x)dx \\ &= \int_0^T \frac{1}{10}e^{-\frac{1}{10}x}dx \\ &= 1 - e^{-\frac{1}{10}T} \end{aligned}$$

Solving for T we get that $T = -10\ln\frac{1}{2} \simeq 6.93$ minutes.

8. (12 points) Beginning in 1749 the Bank of England issued securities known as *British consols* which pay the owner or his heirs a fixed amount of money each year forever. British consols can still be bought and sold in today's securities market.

(a) What is the present value (in British pounds) of a British consol that pays 10 pounds per year? Assume that the first payment is one year from the date of purchase and that the annual interest rate is 5% per year. (Show your work.)

The present value of the payment one year from now is $10/(1+.05)$ *pounds. The present value of the payment two years from now is* $10/(1+.05)^2$ *pounds. The present value of the payment* n *years from now is* $10/(1+.05)^n$ *pounds. Therefore, the present value of this stream of income payments is*

$$PV = \frac{10}{1.05} + \frac{10}{1.05^2} + \cdots + \frac{10}{1.05^n} + \cdots = \frac{10}{1.05}\left(1 + \frac{1}{1.05} + \frac{1}{1.05^2} + \cdots + \frac{1}{1.05^n} + \dots\right).$$

Since the sum of the geometric series $1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$ *when* $|x| < 1$*, taking* $x = 1/1.05$ *we see that*

$$PV = \frac{10}{1.05}\frac{1}{1-\frac{1}{1.05}} = \frac{10}{.05} = 200 \quad \textit{pounds.}$$

(b) Under the same assumptions as in part (a), what is the present value of the first thirty annual payments? (Show your work.)

The solution is the same except we only sum the first thirty terms of the series and then use the formula for the sum of the first thirty terms of a geometric series. That is, the present value is

$$\frac{10}{1.05}\left(\sum_{i=0}^{29} \frac{1}{1.05}^i\right) = \frac{10}{1.05}\left(\frac{1-\left(\frac{1}{1.05}\right)^{30}}{1-\frac{1}{1.05}}\right) \simeq \frac{10(.76862)}{.05} = 153.72 \quad \textit{pounds.}$$

(c) Suppose that, instead of annual payments, the payments are made as a continuous income stream at a constant rate of 10 pounds per year. Assume the interest rate remains at 5% but is now continuously compounded. What then would be the present value of the consol? (Show your work.)

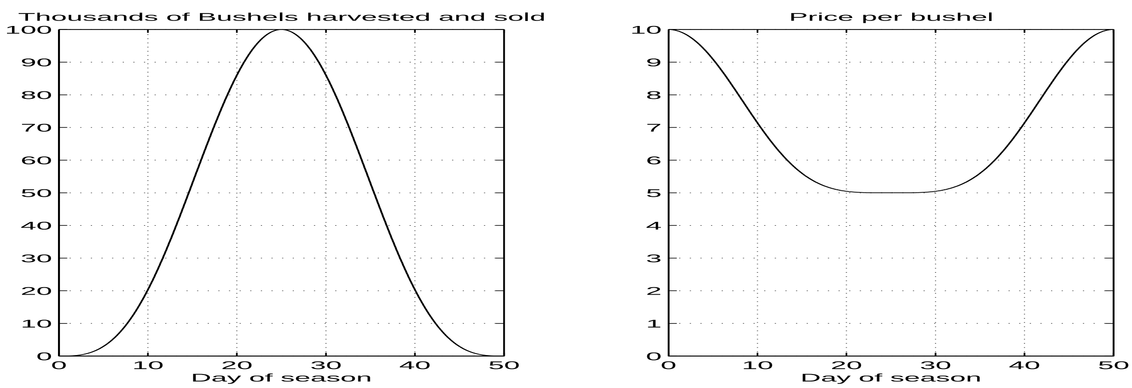
A continuous income stream at a rate of 10 pounds per year receives $10\Delta t$ *pounds between time* t *and* $t + \Delta t$*. The present value of this money discounted at a continuous interest rate of* $r = .05$ *is then* $(10\Delta t)e^{-.05t}$ *pounds. Summing this up over all future times, the present value is found to equal*

$$\int_0^\infty 10e^{-.05t}\,dt = \lim_{b\to\infty} 10\int_0^b e^{-.05t}\,dt = 10\lim_{b\to\infty} -\frac{1}{.05}e^{-.05t}\Big|_0^b = \frac{10}{.05} = 200 \quad \textit{pounds.}$$

9. (10 points) The finance department of Giant Corporate Farms is forecasting their returns from next season's tomato crop. During the fifty-day harvesting season, they predict being able to harvest $B(t)$ thousand bushels of tomatoes per day and sell them at a price of $P(t)$ dollars per bushel on the t-th day after the beginning of the harvest. They estimate that $B(t)$ and $P(t)$ are given by the following functions whose graphs are shown in the figure.

$B(t) = 100\sin^2(\pi t/50)$ bushels (1,000's) per day $\quad P(t) = 5 + 5\cos^4(\pi t/50)$ dollars per bushel.

The price of tomatoes drops rapidly as the number available for sale increases so while Giant Farms begins the season selling tomatoes for \$10 per bushel, at the height of the harvesting season they receive only \$5 per bushel.



(a) Approximately how much money does Giant Farms predict they will receive on day t of the season?

$1000B(t)$ *bushels per day* $\times$ *one day* $\times$ $P(t)$ *dollars per bushel, or*

$$A(t) = \left(100,000\sin^2(\pi t/50)\right)\left(5 + 5\cos^4(\pi t/50)\right) \quad \textit{dollars per day.}$$

(b) Set up an integral that is equal to the amount of money that Giant Farms expects to receive in total for the 50 day harvest. Then compute this amount of money by evaluating the integral (any method allowed, including use of your calculator). Be sure to explain how you obtained the answer.

The total amount of money received between time t and $t + \Delta t$ is approximately $A(t)\Delta t$. Summing this over the 50 day harvest, and then taking the limit as $\Delta t \to 0$, we see that the total amount of money received is

$$\int_0^{50} A(t)\,dt = 100,000\int_0^{50} \sin^2(\pi t/50)\left(5 + 5\cos^4(\pi t/50)\right)\,dt = 14,062,500 \quad \textit{dollars.}$$

The integral was computed numerically by entering the formula for the function $A(t)$ in a calculator and using the numerical integration function.

Continued on next page

(c) Explain how to calculate the average price per bushel that Giant Farms will receive for their tomatoes.

The total number amount of tomatoes sold by Giant Farms is $1000\int_0^{50} B(t)\,dt = 2,500,000$ *bushels. The average price received per bushel is equal to the total amount of money received divided by the number of bushels sold or*

$$\frac{14,062,500}{1000\int_0^{50} B(t)\,dt} = \frac{14,062,500}{2,500,000} = 5.625 \quad \textit{dollars per bushel.}$$

10. (6 points) Explain how Taylor polynomials can be viewed as generalizations of linear approximation. (A good answer to this problem could begin by discussing a special case such as $P_2(x)$, illustrating your points with graphs and equations.)

The tangent line approximation to a function f *near a point* $x = a$ *is given by the first degree polynomial* $P_1(x) = f(a) + f'(a)(x - a)$ *whose graph, a straight line, passes through through the point* $(a, f(a))$ *and has slope equal to* $f'(a)$*. The Taylor polynomial* P_n *of degree of degree* $n > 1$ *that approximates* f *near a point* a *refines this idea by matching not only the function and first derivative values but also all the derivatives of order up to* n*. That is,*

$$f^{(j)}(a) = P_n^{(j)}(a), \quad 0 \le j \le n.$$

As n *increases, the Taylor polynomials* P_n *are expected to give better and better approximations to* f *for* x *near* a*, as illustrated in the following figure which shows the graph of a function, in this case* $f(x) = \ln x$ *and it's Taylor polynomials of degree 1,2, and 5 near the point* $a = 1$*.*