MATH 116 — FINAL EXAM SOLUTIONS

December 15, 2003

NAME: ____

INSTRUCTOR: ___________________ SECTION NO: _________________________

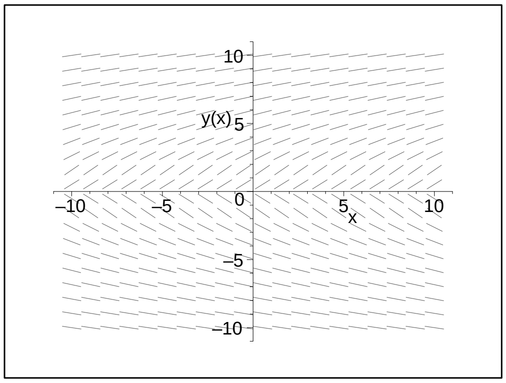
1. Do not open this exam until you are told to begin.
2. This exam has 12 pages including this cover. There are 10 problems.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	6	
2	8	
3	10	
4	8	
5	12	
6	14	
7	10	
8	12	
9	15	
10	5	
TOTAL	100	

1. (6 points) Which of the following differential equations has the slope field given in the figure? (Circle the letter of each correct answer.)

a. $\frac{dy}{dx} = \frac{2x}{1+x^2}$ b. $\frac{dy}{dx} = e^{-y^2}$ c. $\frac{dy}{dx} = \frac{2x^2}{1+x^4}$

d. $\frac{dy}{dx} = \frac{2y}{1+y^2}$ e. $\frac{dy}{dx} = e^{-x^2}$ f. $\frac{dy}{dx} = \frac{2y^2}{1+y^4}$



2. (8 points) Circle "True" or "False" for each of the following statements. No explanation is necessary. (Remember that "True" means the statement is always true.)

(a) The function $y(t) = 0$ is an equilibrium solution of the differential equation $dy/dt = y + t$.

True. **False**.

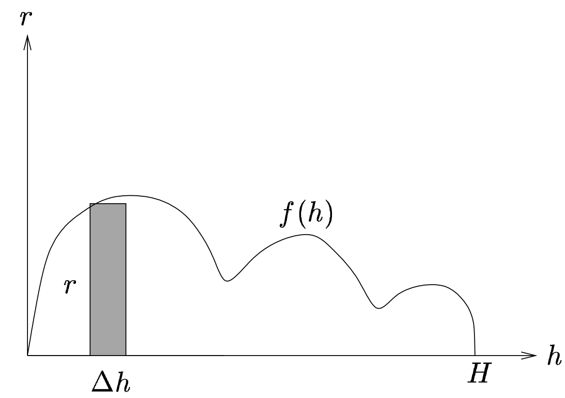
There is no constant y_0 such that $dy/dt - y + t = 0$ for $y = y_0$ and all t, so there is no equilibrium solution of the equation.

(b) If $P(t)$ is a solution of the logistic differential equation, $dP/dt = .5P(200 - P)$, then so is the function $2P(t)$.

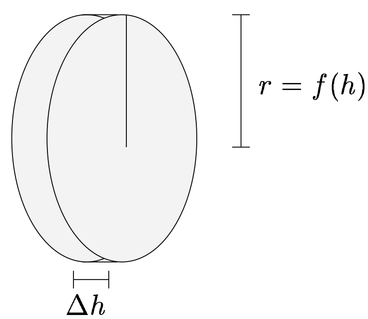
True. **False**.

The constant function $P(t) = 200$ is a solution of the equation, but $2P(t)$ is not.

3. (10 points) The cross-sections of a snowman are given by circles of radius $r = f(h)$ where h is the height measured from the ground and $f(h)$ has graph given in the figure. Both r and h are measured in inches.



(a) Draw and label a typical, thin, cross-section of the snowman. What is the volume of the cross-section (in terms of the function $f(h)$)?



The volume of a typical cross-section can now be read directly from the picture to be

$$V_{\text{slice}} = \pi r^2 \Delta h = \pi [f(h)]^2 \Delta h.$$

(b) Write an integral in terms of $f(h)$ whose value is the total amount of snow used in making the snowman.

To find the total amount of snow used to make the snowman, we add up the volumes of all of the slices forming a Riemann sum, and then let $\Delta h \to 0$. The resulting integral is

$$\int_0^H \pi [f(h)]^2 \, dh.$$

4. (8 points) **(a)** Give the Taylor series about the point $t = 0$ of the function

$$f(t) = \frac{\sin t}{t}.$$

(You are allowed to use the standard Taylor series expansions without deriving them).

We know that

$$\sin(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}$$

so dividing by t gives

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{6} + \frac{t^4}{120} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}.$$

(b) Is the following statement "True" or "False"? Explain why, if true, or why not, if false.

The Taylor series about the point $x = 0$ of the sine integral *function, defined by*

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt,$$

is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots.$$

[True.] False.

The series in part (a) can be integrated term by term to get the Taylor series of the Sine Integral function. That is

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt = \int_0^x 1\, dt - \int_0^x \frac{t^2}{6}\, dt + \int_0^x \frac{t^4}{120}\, dt - \cdots = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}.$$

5. (12 points) **(a)** Carry out two steps of Euler's method with $\Delta x = 0.1$ to approximate the solution $y(x)$ of the initial value problem,

$$\frac{dy}{dx} = xy + y^3, \qquad y(2) = 1.$$

at $x = 2.2$. To receive credit, you must write out the calculations that are needed to make the computation. (No credit for answers alone, even correct ones.)

x	y	$\Delta y = (\text{slope})\,\Delta x = (xy + y^3)\,\Delta x$	$y(x + \Delta x) \approx y(x) + \Delta y$
2	1	$0.3 = (2 \cdot 1 + 1^3)\,0.1$	$1.3 = 1 + .3$
2.1	1.3	$0.493 = (2.1 \cdot 1.3 + 1.3^3)\,0.1$	$1.793 = 1.3 + .493$
2.2	1.793		

So our Euler's method approximation gives $y(2.2) \approx 1.793$.

(b) Will the solution be an increasing function of x for $x \geq 2$? Explain.

Observe that for $x = 2$, we know that $y = 1$ so that $\frac{dy}{dx} > 0$. Therefore the solution is an increasing function at this point. One can then see that for $x > 2$, $\frac{dy}{dx} > 0$ so that the solution is an increasing function on $x \geq 0$.

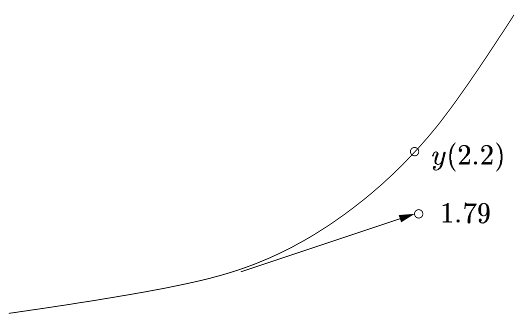
(c) Will your estimate of $y(2.2)$ be an underestimate or an overestimate? Explain.

The solution must be concave up because its second derivative is positive:

$$\frac{d^2y}{dx^2} = y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} > 0$$

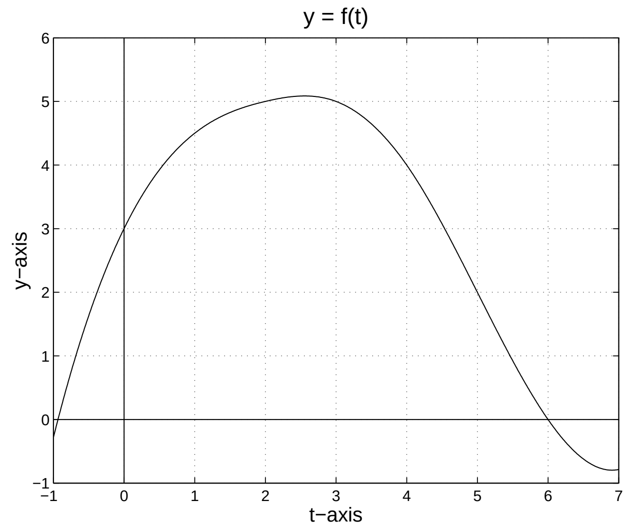
from part (b). Now since Euler's method is a tangent line approximation, we get that 1.79 *is an underestimate of* $y(2.2)$.

Alternatively, note that by part (b), any solution $y(x)$ *of the equation with* $y(x_0) > 0$ *and* $x_0 > 0$ *must be increasing. Therefore,* $xy(x) + y(x)^3$ *is increasing for* $x \geq x_0 > 0$, *and so* dy/dx *is increasing and, consequently,* $y(x)$ *is concave up. As in the first solution, Euler's method, the tangent line approximation, is an underestimate.*



6. (14 points) The function f is defined for $-1 \le t \le 7$ and has graph given in the figure below. The function F is defined by

$$F(x) = \int_2^x f(t)\,dt.$$



(a) Fill in the following table of values of $F(x)$ and $F'(x)$, using the best approximation to the values of these functions that you can determine using the given graph of f.

x	0	2	4	6
$F(x)$	-8.8	0	9.6	13.6
$F'(x)$	3	5	4	0

(b). Compute $g'(2)$, where g is the function defined by $g(x) = F(x^2)$. (Show your work.)

$$g'(x) = F'(x^2)\frac{d}{dx}x^2 = F'(x^2)(2x)$$
$$g'(2) = F'(2^2)(2 \cdot 2) = 4F'(4) = 16.$$

(c) On which subintervals (approximately), if any, of $-1 \le t \le 7$ is F concave up?

F is concave up where $F' = f$ is increasing, which is on the interval $-1 \le t \le 2.8$ (approximately).

7. (10 points) During the holiday season there are two main groups of people at the mall (excluding the store employees). These are the shoppers and the volunteers ringing bells to collect money for charities. The numbers of each vary over time. If we let $B(t)$ be the number of bell ringers at time t and $S(t)$ be the number of shoppers at time t, and assume these are modeled by a predator-prey system of differential equations, then the differential equations describing their numbers are

$$\begin{aligned} \frac{dB}{dt} &= -1,000\,B + 2\,BS \\ \frac{dS}{dt} &= 66\,S - 11\,BS. \end{aligned}$$

(a) Given this model, which is the "predator" and which is the "prey"? Make sure you justify your answer by explaining how this is reflected in the given equations.

The bellringers, B, are the predators since their population in increased by the interaction between the two species, (i.e. the $+2\,BS$ term), while the population of shoppers, S is decreased by the interaction term $(-11\,BS)$.

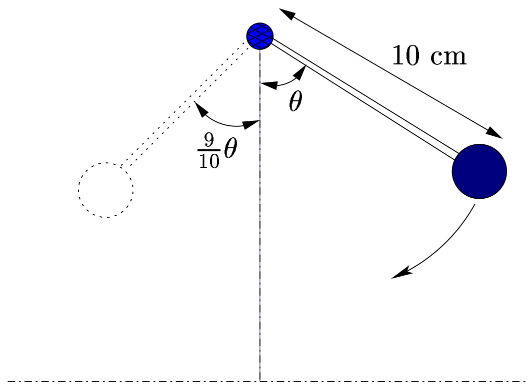
(b) What are the equilibrium points of this system? Describe what the equilibrium points mean in terms of this problem.

The equilibrium points are where the poputlation of shoppers and bell ringers does not change with time, or $dB/dt = dS/dt = 0$. That is,

$$\begin{aligned} -1000B + 2\,BS = B(-1000B + 2S) = 0 \\ 66S - 11BS = S(66 - 11S) = 0 \end{aligned}$$

which has two solutions, namely $(B, S) = (0, 0)$ and $(B, S) = (6, 500)$. The first equilibrium solution represents the situation where there are no bell ringers or shoppers. The second means that, according to this model, when there are 6 bell ringers and 500 shoppers, the number of bell ringers and shoppers will remain constant.

8. (12 points) You begin a pendulum swinging in the position shown in the figure below with $\theta = \pi/4$. Assume the pendulum travels in a circular arc, swinging to the left past the center line shown in the figure and then returning to the right. Notice that the pendulum must briefly stop its motion before it can change direction. We define one "swing" of the pendulum to be the motion between the times when the pendulum stops its motion to change direction. For example, the first "swing" is the motion from the time you release the pendulum until it swings all the way to the left. The second "swing" is the motion coming back from the left to the right, and so on.



(a) Assume that at the end of each swing the pendulum makes an angle of $\frac{9}{10}$ the angle it made when it began the swing. What angle does the pendulum make after its second swing? After its third swing? After its n^{th} swing?

If $\theta_j > 0$ is the angle made by the pendulum with the vertical by the pendulum after the j-th swing, then $\theta_0 = \pi/4$ (given), $\theta_1 = .9\pi/4$, $\theta_2 = .9\theta_1 = (.9)^2\pi/4$, and in general $\theta_{j+1} = .9\theta_j$. Therefore,

$$\theta_2 = (.9)^2\frac{\pi}{4}, \qquad \theta_3 = (.9)^3\frac{\pi}{4}, \qquad \theta_n = (.9)^n\frac{\pi}{4}.$$

(b) Recall that the arc length of a circle is given by the formula $s = r\alpha$ where s is arc length, r is the radius of the circle, and α is the angle measuring the arc length. How far does the weight travel on its first swing? On its second swing? On its n^{th} swing?

If S_j = distance travelled on the j-th swing, then

$$S_1 = \textit{radius} \times \textit{angle} = 10(\theta_0 + \theta_1) = 10\left(\frac{\pi}{4} + (.9)\frac{\pi}{4}\right) = 19\frac{\pi}{4}\ cm$$
$$S_2 = 10(\theta_1 + \theta_2) = 10\left((.9)\frac{\pi}{4} + (.9)^2\frac{\pi}{4}\right) = (.9)S_1\ cm$$
$$S_n = 10(\theta_{n-1} + \theta_n) = 10\left((.9)^{n-1}\frac{\pi}{4} + (.9)^n\frac{\pi}{4}\right) = (.9)\left((.9)^{n-2}\frac{\pi}{4} + (.9)^{n-1}\frac{\pi}{4}\right) = (.9)S_{n-1}$$
$$= (.9)^2S_{n-2} = (.9)^3S_{n-3} = \ldots (.9)^{n-1}S_1 = (.9)^{n-1}(19)\frac{\pi}{4}\ cm$$

Problem continued on next page.

(c) What is the total distance the weight has travelled after 30 swings?

Using part (b), this number is

$$S_1 + S_2 + \dots S_{30} = S_1 + (.9)S_1 + (.9)^2 S_1 + \cdots + (.9)^{29} S_1 = S_1 \left(1 + (.9) + (.9)^2 + (.9)^{29}\right)$$
$$= S_1 \frac{1 - (.9)^{30}}{1 - .9} = 10S_1(1 - (.9)^{30})$$

where the next to last equality is obtained using the formula for the sum of a finite geometric series, $1 + x + x^2 + \cdots + x^{n-1} = (1 - x^n)/(1 - x)$. *Using a calculator to compute* $1 - (.9)^{30})$ *and* $S_1 = 19\frac{\pi}{4}$, *we find that the distance travelled in 30 swings is, correct to the digits shown,* $\left(190\frac{\pi}{4}\right) .9576 = 142.8998$ *cm.*

(d) If the pendulum were allowed to swing forever how far would it travel?

This is similar to part (c) except that we must compute the sum of the infinite geometric series, for which we have a known formula, $1 + x + x^2 + \cdots = 1/(1 - x)$ *whenever* $|x| < 1$.

$$S_1 + S_2 + S_3 + \cdots + S_n + \dots = S_1 \left(1 + (.9) + (.9)^2 + (.9)^3 + \cdots + (.9)^n + \dots\right)$$
$$= S_1 \frac{1}{1 - .9} = 10S_1 = 190\frac{\pi}{4} \approx 149.2257 \, cm.$$

More than 95% of the total distance travelled would be taken up in the first 30 swings.

9. (15 points) Frodo Baggins of the Shire is given the task of taking the ring of power from the elven kingdom of Rivendell to Mount Doom, 100 km away, to destroy it. The ring's weight w (in kg) grows at a rate of one-one-hundredth (1/100) of Frodo's distance x (in km) from Rivendell as Frodo proceeds on his journey. Frodo can travel toward Mount Doom at the rate of 2.5 km per hour except that the weight of the ring of power slows his rate of travel (in km/hr) to Mount Doom by one-twentieth (1/20) of the weight of the ring (in kg). Suppose that the ring weighs .001 kg (1 gram) when he begins his journey from Rivendell.

(a) Write a pair of differential equations that model this situation. What are the initial conditions at time $t = 0$?

With $w(t)$, the weight of the ring t hours after leaving Rivendell and $x(t)$, the distance from Rivendell on the road to Mound Doom,

$$\frac{dw}{dt} = \frac{x}{100}, \qquad \frac{dx}{dt} = 2.5 - \frac{w}{20}$$
$$w(0) = .001 \qquad x(0) = 0$$

(b) What differential equation models the relationship between the weight of the ring and the distance from Rivendell?

$$\frac{dx}{dw} = \frac{\frac{dx}{dt}}{\frac{dw}{dt}} = \frac{2.5 - \frac{w}{20}}{\frac{x}{100}} = \frac{250 - 5w}{x}.$$

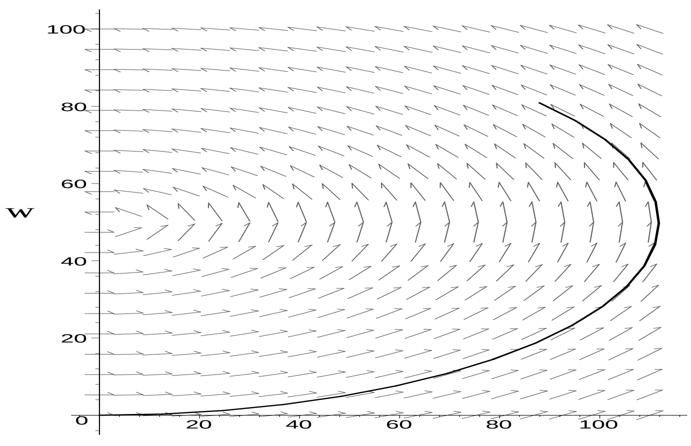
(c) Find the trajectory that Frodo follows in the x-w phase plane by solving the differential equation you found in part (b). (Show your work. Don't forget the initial conditions from part (a)).

We solve the differential equation in part (b) for the trajectories of the solutions by separation of variables. One has $x\,dx = (250 - 5w)dw$. Integrate to get $x^2/2 = (250 - 5w)^2/(-10) + C$ where C is a constant of integration. Simplifying the equation then gives

$$x^2 + 5(50 - w)^2 = A.$$

where $A = 2C$. Since $x = 0, w = .001$ is a point on the trajectory, we can find the constant A by substituting these values into the equation. We find $A = 5(50 - .001)^2 \simeq 5(50)^2$. so the trajectory is

$$x^2 + 5(50 - w)^2 = 5(49.999)^2.$$



(d) Will Frodo complete his task by traversing the 100 km. from Rivendell to Mount Doom or will he fall short of accomplishing this goal? Explain why or why not. (Hint: Frodo is stopped if $\frac{dx}{dt} = 0$, i.e., the weight of the ring becomes too much for him.)

The trajectory is an ellipse with center at $x = 0$, $w = 50$ so rhe maximum value of x on this ellipse occurs when $w = 50$, the point at which $dx/dt = 0$. At this value of w, we find from the equation that $x = (50 - .001)\sqrt{5} \simeq 111.8$. So, Frodo will make the 100 km to Mount Doom.

10. (5 points) Suppose that on your visit home over break you meet a friend who is now taking precalculus at your old high school. He knows the formula "distance travelled = rate × time." He also knows some students who are taking the calculus course at the high school, and he has heard there is a more general formula, "distance travelled = area under the velocity curve," that computes the distance, even when the velocity is not constant. He asked those students to explain this second formula, but they just shrugged and said he would have to wait until he learned calculus to get an explanation.

Write down what you would tell your friend to explain why the second formula holds and how it is related to the formula he has learned in precalculus. Be sure to include any appropriately labelled graphs you might draw in making your explanation.

The precalculus formula of $d = v\,t$ gives the formula relating the distance, time and velocity when the velocity is constant over time. One can think of this as the area under a velocity curve given by a horizontal line, as can be seen in (Fig. 1).

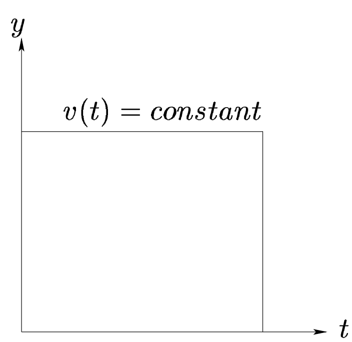
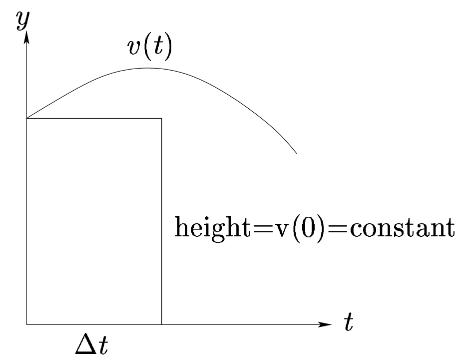


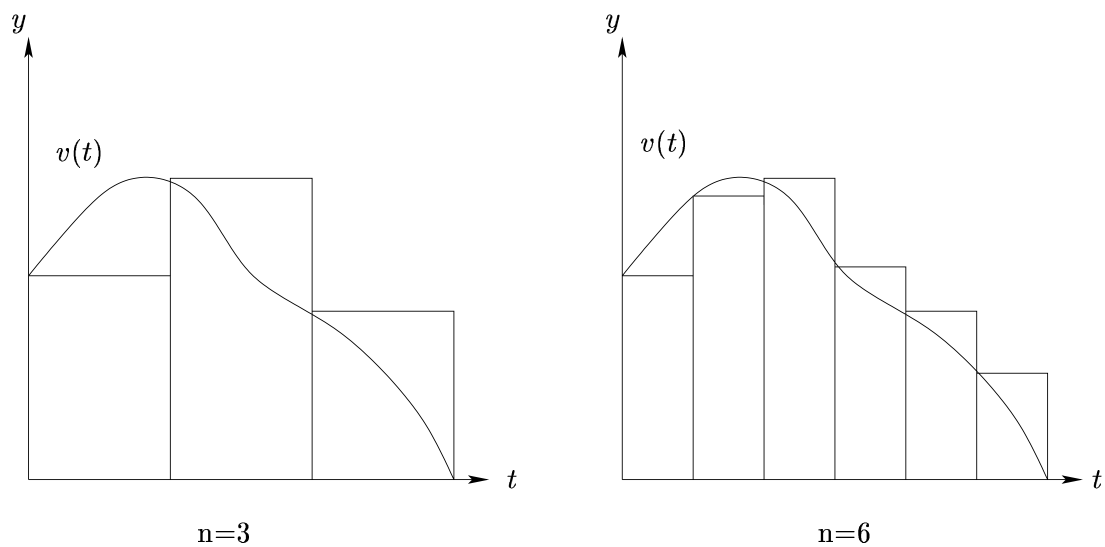
*Fig.*1

However, when the velocity function varies with time, we can approximate it on small intervals to be a constant function. The height of the rectangle gives an approximation to the actual velocity function. Therefore the actual distance travelled over this small interval is approximately the distance travelled by the constant velocity we are approximating the actual velocity by. So the actual distance is approximately the height of the rectangle × the time interval we are using. Note that this is actually the area of the rectangle! See (Fig. 2).



$Fig. 2$

Now to get an approximation of the total distance travelled, one merely adds up the various rectangles used in the approximation. As the time interval used decreases to 0, the number of rectangles used increases to infinity and we see that the area of the rectangles actually fills out the entire area under the velocity curve. See (Fig. 3) for the cases of 3 rectangles and 6 rectangles.



$Fig. 3$

This is how one goes from the simple precalculus formula of distance = rate × time to the area under the velocity curve being the distance travelled.