MATH 116 — SECOND MIDTERM EXAM

Fall 2004

NAME: ____________________________ ID NUMBER: ___________________________

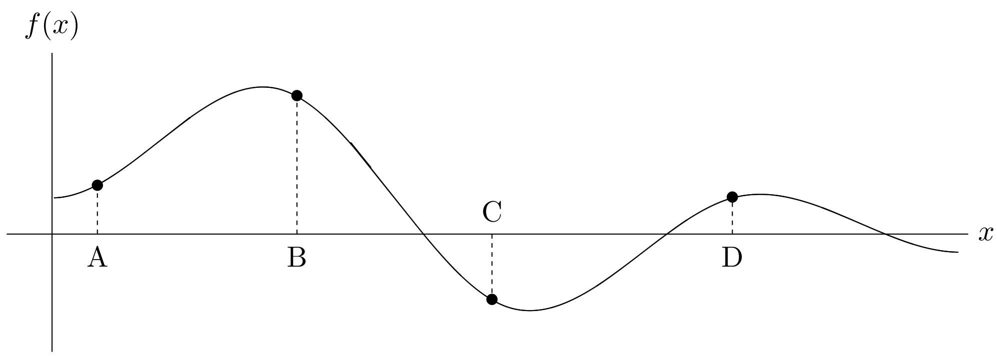
INSTRUCTOR: ____________________ SECTION NO: ___________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and devices whose sounds might disturb your classmates. Please remove **all** headphones.

PROBLEM	POINTS	SCORE
1	16	
2	10	
3	10	
4	18	
5	10	
6	12	
7	11	
8	13	
TOTAL	100	

1. (16 points) For each of the three questions, fill in the blank(s) using the appropriate suggested answer(s). *No explanation is required.*

(a) The polynomial $P_2(x) = 1 + 3(x-a) - 2(x-a)^2$ is the second degree Taylor polynomial approximating the function f for x near a. The graph of f is given in the figure. Which of the points A, B, C, or D on the x-axis has a as its x-coordinate?



ANSWER : ____________ .

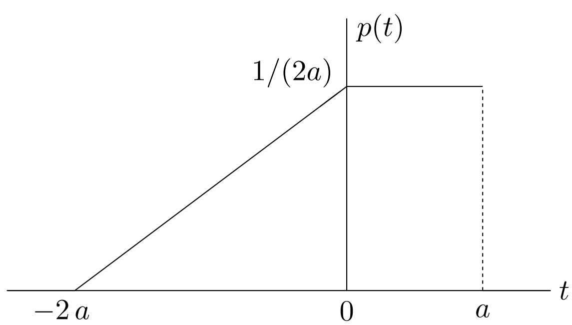
(b) Three of the tests for deciding the convergence or divergence of an infinite series are:

A. integral test, **B.** comparison test, **C.** ratio test.

Using each of these letters **A, B, C** *exactly once*, fill in the blank by each of the following infinite series with the label of the most appropriate test to use in deciding whether the series converges or diverges.

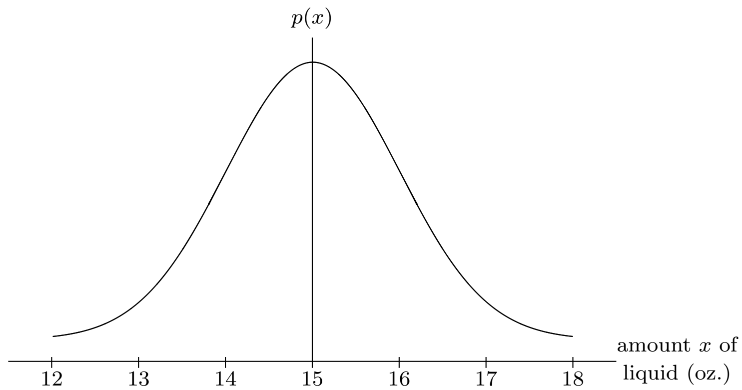
$\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ____________ $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin^2 n}{1+n^{5/2}}$ ____________ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ ____________

(c) The graph of the distribution $p(t)$ is shown on the figure, where $a > 0$ is a constant. Fill in the blank with "*greater than*", "*equal to*", or "*smaller than*" to make the sentence below the graph correct.



The median of the distribution p(t) is ______________ its mean.

2. (10 points) A firm that manufactures and bottles apple juice has a machine that automatically fills bottles with 15 ounces (oz) of apple juice. There is some variation, however, in the amount of liquid dispensed in each bottle. Over a long period of time, the average amount dispensed into the bottles was 15 ounces, but the underlying measurements showed the distribution of the ounces, x, of juice in the bottles was given by $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-15)^2}$.



(a) What fraction of the bottles contained between 14 and 16 oz of juice? *Explain.*

(b) Give a graphical interpretation of your answer to part **(a)** on the figure.

(c) Find, as accurately as you can, the fraction of the bottles that contained at least 17 oz of juice inside them. *Explain.*

3. (10 points)

(a) Find the radius of convergence R of the following power series. *Show your work.*

$$\sum_{n=1}^{\infty} \frac{(n + n^3\, 2^n)}{n^2\, 3^n}\,(x-1)^n\,.$$

ANSWER : $R\ =$ ______________.

(b) What is the interval of convergence of the series?

$$\_\_\_\_\_\_\_\_\_\_ < \quad x \quad < \_\_\_\_\_\_\_\_\_\_.$$

4. (18 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**. *You need not explain your answer.*

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

T **F**

(b) The Taylor series for $f(x)g(x)$ near $x = 0$ is $f(0)g(0) + f'(0)g'(0)x + \frac{1}{2}f''(0)g''(0)x^2 + \cdots$.

T **F**

(c) If $p(x)$ is the probability density function of some characteristic x distributed throughout a population, then $p(3) = 0.4$ means that 40% of the population has $x < 3$.

T **F**

(d) The infinite series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ converges.

T **F**

(e) If the Taylor polynomial of degree four of $f(x)$ about $x = 0$ is $2 - 3x + 5x^3 + 7x^4$, then the Taylor polynomial of $g(x) = \frac{f(x^2) - 2}{x}$ of degree five about $x = 0$ is $5x^5 - 3x$.

T **F**

(f) Let $P(x)$ be the cumulative distribution function of the blood cholesterol level of men aged 40 and over in the US population, measured in milligrams (mg) per deciliter (dL). The equality $P(190) = 0.5$ means that the median blood cholesterol level of men in this population is 190 mg/dL.

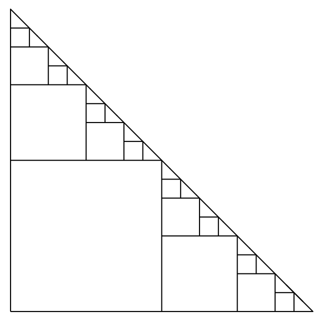
T **F**

5. (10 points)
(a) Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0. *You must show the calculations that lead to your answer.*

(b) What is the Taylor series about $x = 0$ of the function $\sin x$? *No explanation required.*

(c) Using your answers to parts **(a)** and **(b)** and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4 + \sin(2x)}$ for x near 0. *Show your work.*

6. (12 points) We have learned how to use slicing to calculate areas and volumes. This problem explores a different kind of slicing through a simple example. A right-isoceles triangle with sides of length 2 is covered by squares as illustrated and explained in the figure below.



step 1: one square of side length 1

step 2: two squares of side length 1/2

step 3: four squares of side length 1/4

step 4: eight squares of side length 1/8

... etc ...

(a) Use a geometric series to find the area covered by the squares after the N^{th} step.

(b) Use your answer to part **(a)** and your knowledge of series to find the total area covered by the infinitely many squares.

(c) How do you know your answer to part **(b)** is the correct one?

7. (11 points) Suppose that as you finish your degree at Michigan, you develop a great idea for starting a company that, you believe, is dead certain to start out making a profit at a rate of \$40,000 per year (after your salary and all expenses). Further, your idea is so good that the profits will increase in future years so that t years after starting the company, it will be making profits at a rate of $40,000 + 10,000t$ dollars per year. Assuming your projections are correct,

(a) How much profit will your company make in the first 10 years of operation?

(b) What is the value at the time your company is started of the first 10 years of its profits? Assume an interest rate of 8% per year, compounded continuously. Be sure to both write an integral whose value is equal to the present value you are computing and evaluate the integral.

(c) Being a bright and inquisitive person, you'd like to sell your company after a few years to pursue other interests. One common way of assigning a value to a company is that it is worth, "the present value of all its future earnings". Assuming your profit projections are correct, the same interest rate of 8% with continuous compounding as in part (b), and receiving this value for your company when you sell it in 10 years, write an integral whose value is equal to the amount you would receive at that time.

8. (13 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e \,, \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the third order Taylor polynomial for the function $\sqrt{\dfrac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2}$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the Doppler Law for slow-moving emitters:

$$F_r \simeq \left(1 - \frac{v}{c}\right) F_e \,.$$

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 10% of the speed of light c? *Explain.*