MATH 116 — FINAL EXAM

Fall 2004

NAME: ______________________________ ID NUMBER: ____________________________

INSTRUCTOR: _____________________ SECTION NO: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 12 pages including this cover. There are 11 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about the problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and devices whose sounds might disturb your classmates. Please remove **all** headphones.

PROBLEM	POINTS	SCORE
1	16	
2	6	
3	5	
4	6	
5	10	
6	15	
7	5	
8	6	
9	10	
10	8	
11	13	
TOTAL	100	

1. (16 points)

(a) If $\int_0^1 f(x)\,dx = 2$, then $\int_0^2 f\left(\frac{x}{2}\right)\,dx =$ ______________.

(b) Does the infinite series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge? *Justify your answer.*

(c) Is the function te^{-2t} a solution of the differential equation $\frac{dy}{dt} + 2y - e^{-2t} = 0$? *Explain why or why not.*

(d) Suppose $C(t)$ is the daily cost of heating your house, measured in dollars per day, where $t = 0$ corresponds to January 1, 2004. Give the meaning, in words, of each of the following quantities.

(i) $\int_0^{60} C(t)\,dt\,.$

(ii) $\frac{1}{60}\int_0^{60} C(t)\,dt\,.$

2. (6 points) Write a parametrization for each of the following curves in the xy-plane.

(a) The circle of radius 2, centered at the origin, traced clockwise, and starting from $(-2, 0)$ when $t = 0$.

(b) The line passing through the points $(2, -1)$ and $(1, 3)$.

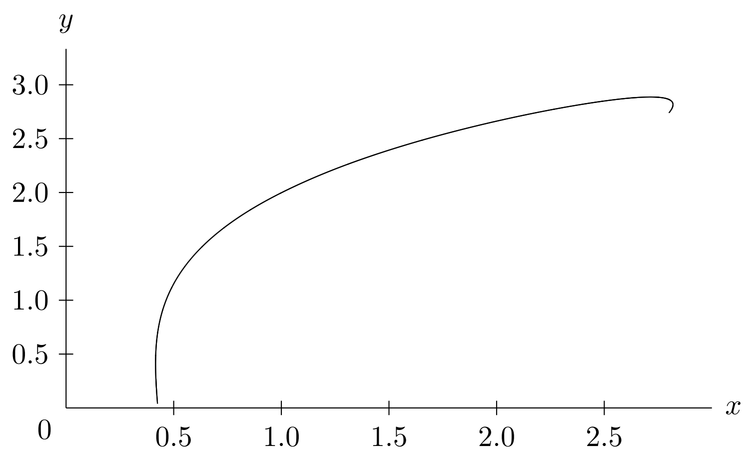
3. (5 points)
(a) Briefly explain the difference between the indefinite integral $\int f(x)\, dx$ and the (proper) definite integral $\int_a^b f(x)\, dx$.

(b) What is a Riemann sum and how is it related to one or more of the integrals in part **(a)**?

4. (6 points) A particle moves in the xy-plane so that it is at the position $(x(t), y(t))$ at time t, where $x(t)$ and $y(t)$ satisfy the system of differential equations

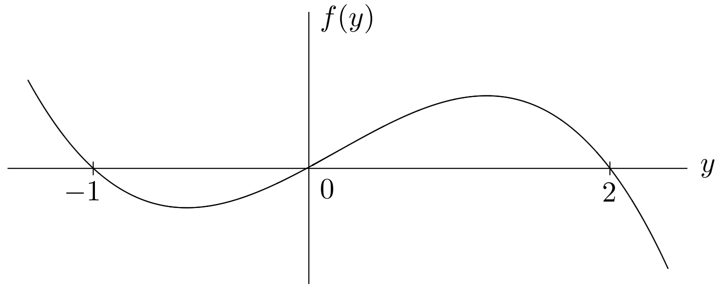
$$\frac{dx}{dt} = x^2 - y^2 \,, \qquad \frac{dy}{dt} = x - 2t \,.$$

It is known that at time $t = 2$, the particle is at the point $(1, 2)$. A graph of the path of the particle is shown in the figure.



Find the instantaneous velocity of the particle at time $t = 2$, and draw an arrow along the curve that shows the direction of motion. *Show your work.*

5. (10 points) Let f be the function whose graph is given on the figure below.



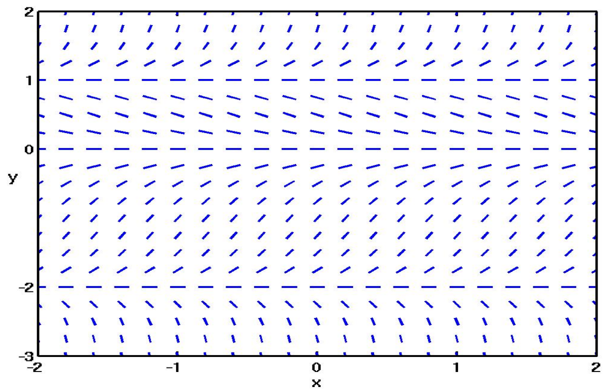
(a) On the **separate yellow sheet**, six field plots are displayed. Choose the one that corresponds to the differential equation $\dfrac{dy}{dx} = f(y)$.

ANSWER : _______________.

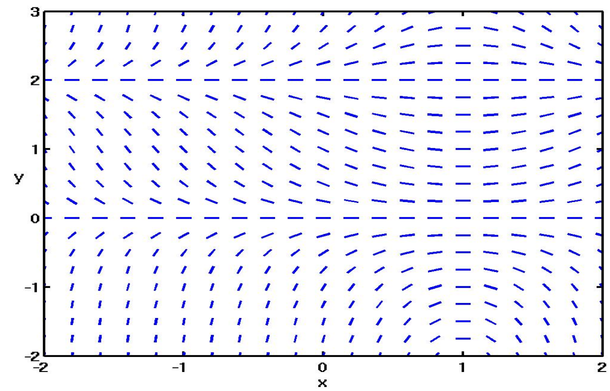
(b) Find all the equilibrium solutions of the differential equation $\dfrac{dy}{dx} = f(y)$.

(c) Which of the equilibrium solutions you found in part **(b)** are stable? *Explain the reason(s) for your answer(s).*

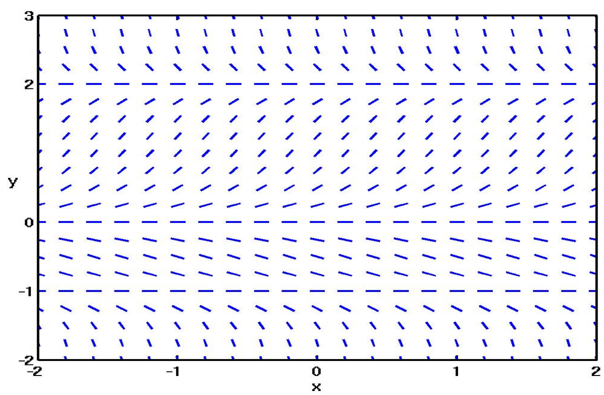
The plots shown on this page refer to Problem 5 on page 5.



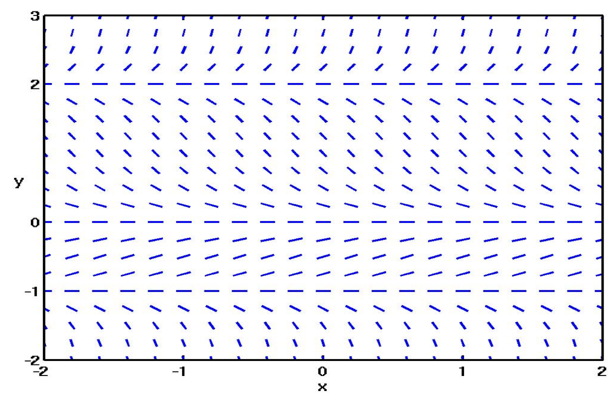
Plot A



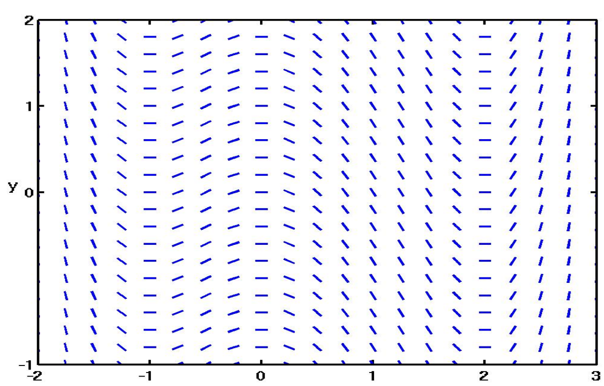
Plot B



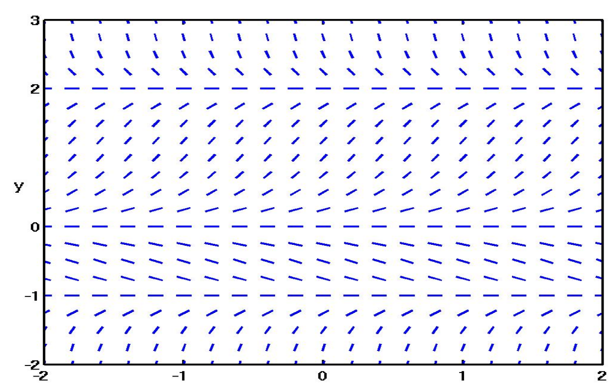
Plot C



Plot D



Plot E



Plot F

6. (15 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**. *You need not explain your answer.*

(a) The formula $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ holds for all real numbers $x \neq 1$ and all positive integers $n = 1, 2, 3, \ldots$.

T **F**

(b) If $g(x)$ is a periodic function, then every solution $y = f(x)$ of the differential equation $\dfrac{dy}{dx} = g(x)$ is also a periodic function.

T **F**

(c) If $y = f(t)$ is a solution of the differential equation $\dfrac{dy}{dt} = y^2 - t$, then for every constant C, $f(t) + C$ is also a solution of the differential equation.

T **F**

(d) The function $y(t) = 0$ is a solution of the initial value problem

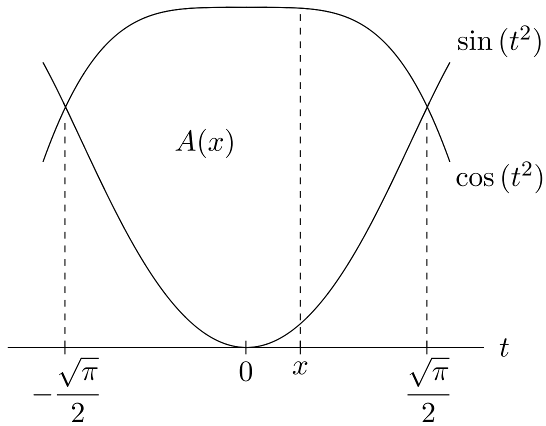
$$\frac{dy}{dt} = 3t - y^3\,, \quad y(0) = 0.$$

T **F**

(e) There is a solution of the logistic differential equation $\dfrac{dP}{dt} = 0.03P\left(1 - \dfrac{P}{3}\right)$ that satisfies $P(1) = 1$ and $P(20) = 5$.

T **F**

7. (5 points) For $-\frac{\sqrt{\pi}}{2} \le x \le \frac{\sqrt{\pi}}{2}$, let $A(x)$ be the area of the region bounded by the curves $\cos(t^2)$, $\sin(t^2)$, and the vertical lines $t = -\frac{\sqrt{\pi}}{2}$ and $t = x$. See the figure below.



(a) Sketch on the figure an area that represents $\Delta A = A(x+\Delta x) - A(x)$ for a small number Δx.

(b) Find a formula for the derivative $A'(x)$.

ANSWER : $A'(x) =$ __________________________.

8. (6 points) For what values of the positive number p does the infinite series $\sum_{n=1}^{\infty} \frac{n^3 - 4n^2}{n^p + 5}$ converge? *Explain the reason for your answer.*

ANSWER : _____________.

9. (10 points) At age 65, Mrs. Smith retires with \$1,000,000 in her retirement account. Assume that after retirement:

(i) She receives interest of 5% per year (compounded continuously) on the balance in the account, and this money is reinvested in the account ;

(ii) She withdraws money (for living expenses) from the account at a continuous rate of \$60,000 per year.

(a) Write the initial value problem for the balance $B(t)$ of dollars in the account t years after Mrs. Smith retires.

(b) Will Mrs. Smith ever exhaust the retirement account, i.e. reduce the balance in the account to zero? *Explain.*

(c) Are there any equilibrium solutions to the differential equation of part **(a)**? If so, explain their meaning in terms of Mrs. Smith's money.

10. (8 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e \ , \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the Taylor series for the function $\sqrt{\dfrac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2} + \cdots$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the approximation to the Doppler Law for slow-moving emitters:

$$F_r \simeq \left(1 - \frac{v}{c}\right) F_e \ .$$

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 20% of the speed of light c? *Explain.*

11. (13 points) In normal conditions, the thyroid hormone (Hormone T), produced in the thyroid gland, and the thyroid successor hormone (Hormone S), produced in the pituitary gland, form a so-called "auto-regulated feedback process". The amount of one in the bloodstream influences the production of the other, and vice-versa. The simple system given below models this process, where x is the amount of Hormone T (in standard units), and y is the amount of Hormone S (in standard units), present in the bloodstream at time t hours.

$$\frac{dx}{dt} = 3 - y\,, \qquad \frac{dy}{dt} = x - 2\,.$$

(a) Find all equilibrium solutions (if any) of the system.

(b) Suppose that at $t = 0$, the amount of Hormone T in the blood was 1.0 and the amount of Hormone S was 3.5, both in standard units. Find the equation of the trajectory of the corresponding solution curve in the phase plane. *Show your work.*

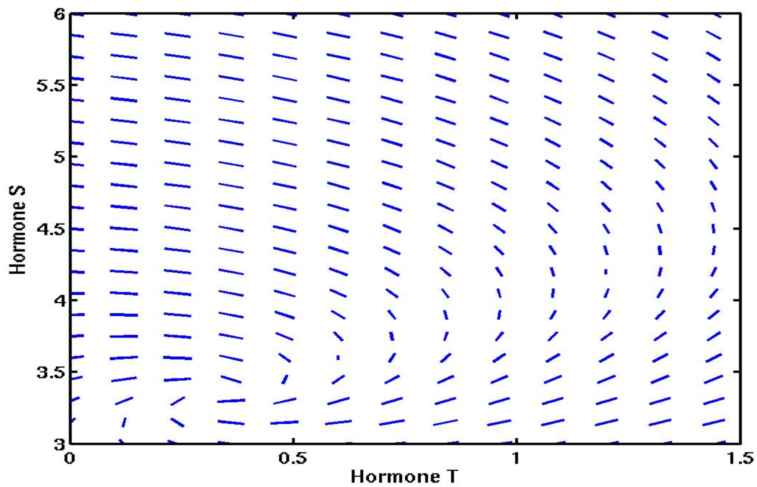
Problem continued on next page.

Problem continued from previous page.

If the patient's diet lacks iodine (e.g. from salt), the chemical agent responsible for detecting the presence of Hormone S in the blood is no longer active. The above model must be replaced by the new system:

$$\frac{dx}{dt} = 3 - y + x\,, \qquad \frac{dy}{dt} = x - 2 + \frac{y}{2}\,.$$

The slope field for the differential equation that describes the trajectories of this system is shown on the figure below.



(c) Sketch on the figure the trajectory corresponding to the initial values in part **(b)**; that is, $x(0) = 1.0$ and $y(0) = 3.5$. *You need not solve any differential equation.*

(d) In the context of this problem, briefly describe how the amounts of the hormones change from their initial values as time increases.