MATH 116 — FIRST MIDTERM EXAM

Fall 2004
Solutions

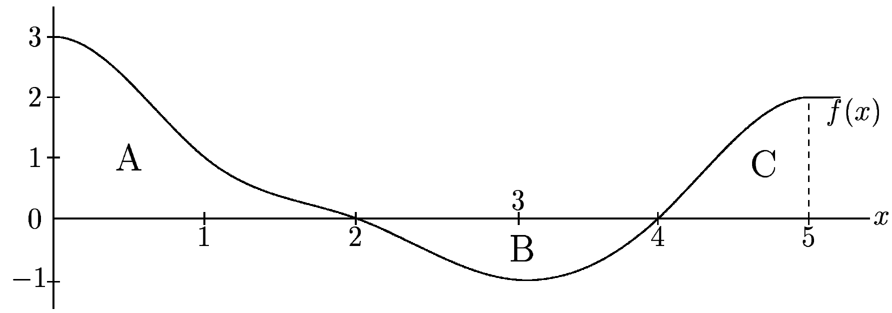
NAME: ______________________________ ID NUMBER: ______________________________

INSTRUCTOR: ____________________ SECTION NO: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and devices whose sounds might disturb your classmates. Please remove **all** headphones.

PROBLEM	POINTS	SCORE
1	15	
2	8	
3	8	
4	10	
5	12	
6	15	
7	18	
8	14	
TOTAL	100	

1. (15 pts.) For $0 \le x \le 5$, let $g(x) = \displaystyle\int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure. The areas of the regions bounded by the graph of f and the x-axis, and labeled A, B, C are equal to 2.5, 1, and 1, respectively.



(a) Find, as accurately as you can...

(i) the values of:

$g(2) = \displaystyle\int_0^2 f(x)dx = \mathbf{2.5}$, $\quad g(4) = \displaystyle\int_0^4 f(x)dx = 2.5 - 1 = \mathbf{1.5}$, $\quad g'(5) = f(5) = \mathbf{2}$

(ii) the interval(s) on which g is decreasing.

From the FTC, we know $g'(x) = f(x)$. Since g is decreasing where g' is negative, we're looking for the intervals where f is negative. There's only one such interval: $\mathbf{2 < x < 4}$.

(iii) the interval(s) on which g is concave up.

Again, from the FTC, we know $g''(x) = f'(x)$. Since g is concave up where g'' is positive, we're looking for the intervals where f' is positive, i.e. the intervals on which f is increasing. From the graph of f, we see there's only one such interval: $\mathbf{3 < x < 5}$.

(iv) the value(s) of x and $g(x)$ for the value(s) of $0 \le x \le 5$ where $g(x)$ is largest.

We know g is increasing where its derivative, f, is positive, i.e. on $0 < x < 2$ and $4 < x < 5$, and decreasing on the interval $2 < x < 4$ where f is negative. Therefore, the largest value of g on the interval $0 \le x \le 4$ must occur at $x = 2$, where we have $g(2) = 2.5$. On the interval $4 \le x \le 5$, the largest value of g must occur at $x = 5$ where $g(5) = 2.5$. Therefore, g attains its maximum value of 2.5 at the two points, $x = 2$ and $x = 5$ of the interval.

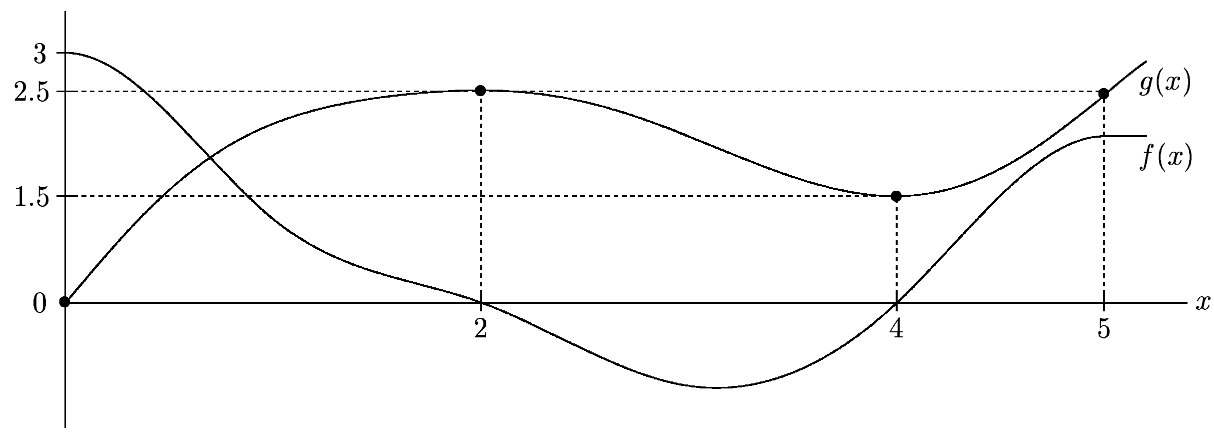
Solution continued on next page.

Solution continued from previous page.

(b) Sketch as accurately as you can the graph of g. Make sure...

- that your graph is consistent with your answers to parts (a)-(d);
- to label any points on the graph where you know the coordinates of the point $(x, g(x))$.

There are only four points on the graph of g which are known exactly: $g(0) = 0$, $g(2) = 2.5$, $g(4) = 1.5$, *and* $g(5) = 2.5$.
We know g is increasing where its derivative, f, is positive, i.e. on $0 < x < 2$ *and* $4 < x < 5$. *Similarly, g is decreasing on* $2 < x < 4$. *Also, as seen in the previous question, g has a local max at* $x = 2$ *and a local min at* $x = 4$.
We know g is concave up where its second derivative, f'*, is positive, i.e. where f is increasing. Thus g is concave up on* $3 < x < 5$. *Similarly, g is concave down where f is decreasing, i.e. on* $0 < x < 3$. *Also, g has inflection points wherever f has a local min or a local max. From the graph, we estimate that happens for* $x = 3$ *only.*
Compiling this information altogether produces a rough sketch of the graph of the function g (see below).



2. (8 pts.) Circle **T** if the phrase completing the sentence is true, or **F** if it is false. You need not explain your answer.

The function $f(x) = \sin(x^2)$

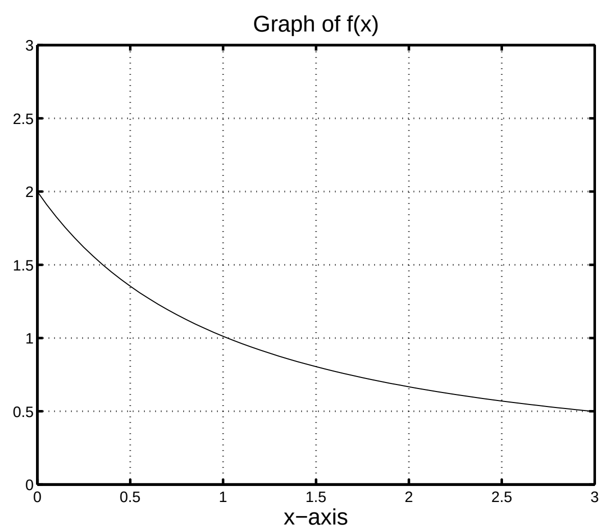
(i) *does not have an antiderivative.* **T** [**F**]

(ii) *has* $-\dfrac{\cos(x^2)}{2x}$ *as an antiderivative.* **T** [**F**]

(iii) *has exactly one antiderivative.* **T** [**F**]

(iv) *has antiderivative* $2x\cos(x^2)$. **T** [**F**]

3. (8 points) For the function whose graph is shown in the figure, the numerical values of the five numbers given by the exact value of the integral (correct to the indicated number of places), the Left Hand Sum, the Right Hand Sum, the Trapezoid Rule, and the Midpoint Rule, with $n = 60$ are listed below. Match them with the corresponding approximation, labeling each number with either EXACT, LHS, RHS, TRAP, or MID. (No explanation is required.)



2.7560 RHS

2.7930 MID

2.7932 EXACT

2.7936 TRAP

2.8311 LHS

4. (10 points) Give the definition of the improper integral $\int_1^{\infty} \frac{1}{x^{3/2}}\, dx$. Then use your definition to evaluate the integral if it converges, or else show it diverges.

The given integral is improper because of the upper limit being infinite. To handle this, we define the improper integral as a limit of definite integrals:

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx \;=\; \lim_{b\to+\infty} \int_1^{b} \frac{1}{x^{3/2}} dx \;=\; \lim_{b\to+\infty} \left[\frac{-2}{x^{1/2}}\right]_1^b \;=\; -2 \lim_{b\to+\infty} \frac{1}{b^{1/2}} + 2 \;=\; 2\,.$$

Therefore the improper integral **converges** *and has the value 2.*

5. (12 points). Evaluate the integrals, given that $f(x)$ is a continuous function for $0 \le x \le 6$ with the following properties:

$$f(0) = 2, \quad f(2) = 3, \quad f(4) = -1, \quad f(6) = 5; \qquad f'(0) = 1, \quad f'(2) = 4;$$

$$\int_0^2 f(x)\, dx = 3, \quad \int_2^4 f(x)\, dx = 1, \quad \int_4^6 f(x)\, dx = 6.$$

(a) $\int_0^2 x f'(x)\, dx \;=\; \mathbf{3}.$

By parts using $u = x$ and $v' = f'$, so $u' = 1$ and $v = f$, we find:

$$\int_0^2 x f'(x) dx \;=\; \left[x f(x)\right]_0^2 - \int_0^2 f(x) dx \;=\; 2 \times 3 - 0 \times 2 - 3 \;=\; 3\,.$$

(b) $\int_2^4 f'(x)\big(2 + 3f(x)\big)\, dx \;=\; \mathbf{-20}.$

Break up the integral into two pieces. The first one is done using the FTC directly, while the second one is handled by noting $f'(x)f(x) = \frac{1}{2}\big(f^2(x)\big)'$.

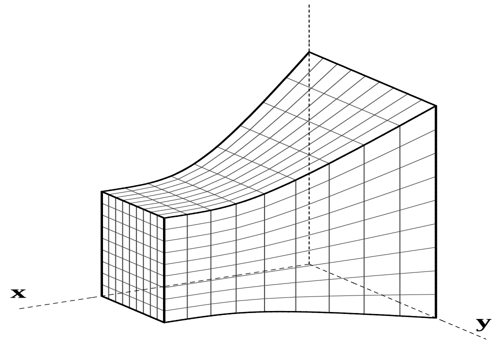
$$\begin{aligned}\int_2^4 f'(x)\big(2 + 3f(x)\big) dx \;&=\; 2\int_2^4 f'(x) dx + \frac{3}{2}\int_2^4 \big(f^2(x)\big)' dx \\ &=\; 2\left(f(4) - f(2)\right) + \frac{3}{2}\big(f(4)^2 - f(2)^2\big) \;=\; -20\,.\end{aligned}$$

(c) $\int_0^2 f(3x)\, dx \;=\; \mathbf{\frac{10}{3}}.$

By substitution, setting $u = 3x$, so $dx = du/3$ and the new limits of integration 0 and 6, we find:

$$\int_0^2 f(3x) dx \;=\; \frac{1}{3}\int_0^6 f(u) du \;=\; \frac{1}{3}\left(\int_0^2 f(u) du + \int_2^4 f(u) du + \int_4^6 f(u) du\right) \;=\; \frac{3+1+6}{3} \;=\; \frac{10}{3}.$$

6. (15 pts.) Consider the region in the x, y plane bounded by the graph of $y = \left(1 + (2 - x)^4\right)^{1/4}$, the x-axis, and the lines $x = 0$ and $x = 2$.



(a) Write an integral giving the value of the volume of the solid whose base is the given region and whose cross-sections perpendicular to the x-axis are squares. (A three-dimensional view of the solid is presented in the figure.)

$$\text{Volume} = \int_0^2 y^2\, dx = \int_0^2 \left(1 + (2 - x)^4\right)^{1/2} dx\,.$$

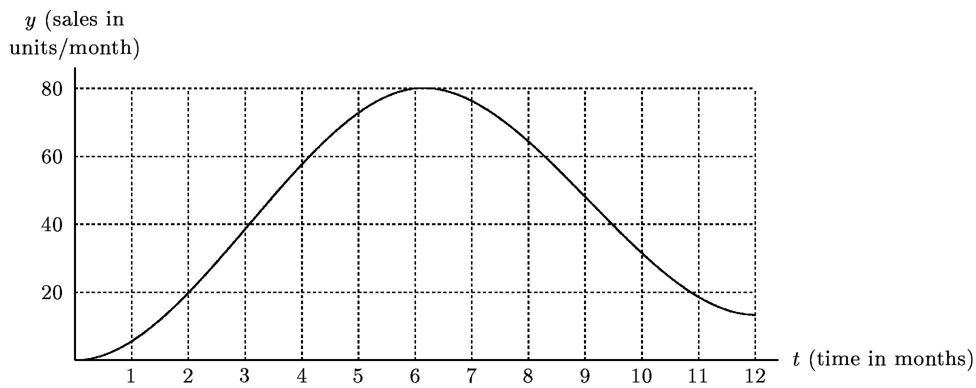
(b) Explain how Riemann sum approximations of your integral are related to approximations of the volume of the solid.

If we slice through the figure with a plane perpendicular to the x-axis that meets the axis at the point $(x, 0, 0)$, then we are given that this slice is a square with side $y = y(x) = \left(1 + (2 - x)^4\right)^{1/4}$. Therefore, it has area equal to $A(x) = y^2 = \left(1+(2-x)^4\right)^{1/2}$. If the solid is sliced into n thin slices by such planes spaced Δx apart, each with thickness equal to Δx, then the volume of the slice through x is approximately equal to $A(x)\Delta x = \left(1 + (2 - x)^4\right)^{1/2}\Delta x$. Adding up these numbers over all slices gives us $\sum_{i=1}^{n} \left(1 + (2 - x_i)^4\right)^{1/2}\Delta x$, which is a Riemann sum approximation to the integral of part (a). It is also a good approximation to the sum of the volumes of the slices, i.e. the volume of the solid. Taking the limit as the number n of slices tends to infinity then gives us that the integral is equal to the volume.

(c) Find, as accurately as you can, the value of the volume of the solid described in part (a). Explain how you computed your answer.

The integral was calculated using the numerical integration function on a TI-83 calculator with the function $y(x) = (1 + (2 - x)^4)^{1/2}$ as integrand and $x = 0$, $x = 2$ as the end points. The volume is approximately **3.65348**.

7. (18 points) Aberister is a small company which sells clothing items for youngsters in North America. When the new designs are released on January 1, 2005, the company projects sales $y(t)$ of shirts (in units of 1,000 items per month) for the year 2005 to be as shown in the figure. As the figure shows, sales are expected to increase until the summer when most "target" customers will have made their purchases, and then decline for the remainder of the year.



(a) Write an expression for $A(t)$, the projected monthly average number of units which will have been sold in the first t months of the year.

$$A(t) = \frac{1}{t}\int_0^t y(s)\,ds \quad \textit{units.}$$

(b) Why is the value of $A(t)$ always less than 80, for any value of $0 < t \leq 12$?

The function $A(t)$ is the average value of $y(t)$ and y is always smaller than 80, so its average is as well. Analyticially, this is because

$$A(t) = \frac{1}{t}\int_0^t y(s)\,ds \leq \frac{1}{t}\int_0^t 80\,ds = 80\,.$$

(c) Approximate the value of $A(t)$ at $t = 3$. Explain how you arrived at your approximation.

The area of each box under the graph of $y(t)$ has the interpretation as 20 units of sales or 20,000 clothing items. Interpreting $A(3)$ as the area under the graph of A from $t = 0$ to $t = 3$, we see this is approximately equal to two and a quarter boxes, or 45 sales units. That is,

$$A(3) = \frac{1}{3}\int_0^3 y(t)\,dt \approx \frac{45}{3} = \mathbf{15} \;\; \textbf{\textit{units.}}$$

(The actual value from the function whose graph is shown in the figure is about 14.46 units)

ANSWER: $A(3) \approx$ 15 units

Solution continued on next page.

Solution continued from previous page.

Assuming the price of the shirts remains constant throughout the year, Aberister will maximize its profit on the 2005 designs by launching its next collection, the holiday seasons designs, when $A(t)$ is maximum.

(d) Suppose that at time t_{max} the value of A is maximum. What is the relationship between the values of y and A at time t_{max}? Explain.

At a maximum value of $A(t)$, we must have $A'(t_{max}) = 0$. By the quotient rule and then the FTC, we have

$$A'(t) = \frac{t\frac{d}{dt}\int_0^t y(s)\,ds - \int_0^t y(s)\,ds}{t^2} = \frac{ty(t) - \int_0^t y(s)\,ds}{t^2}.$$

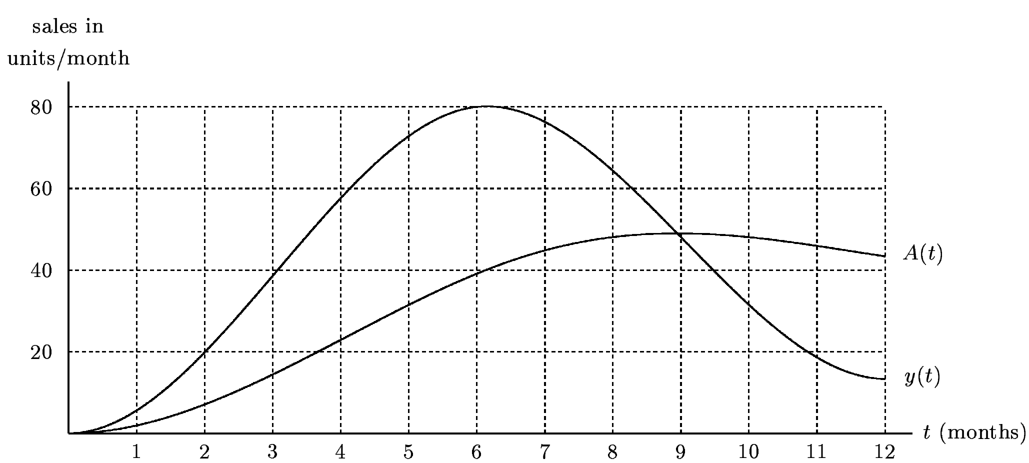
The equation $A'(t_{max}) = 0$ implies that the numerator of this fraction vanishes. Dividing the numerator by t_{max} then gives

$$y(t_{max}) = \frac{1}{t_{max}} \int_0^{t_{max}} y(s)\,ds = A(t_{max}).$$

(e) Sketch a graph of the function $A(t)$ on the above figure (previous page). Be sure to show where $A(t)$ is increasing and decreasing. Use your graph to estimate the time of the year when Aberister should launch its new collection.

The function $A(t)$ will be increasing so long as $y(t) > A(t)$ and decreasing if $y(t) < A(t)$. Therefore, its graph must cross the graph of $y(t)$ at the point where $A(t)$ is a maximum. We also saw this in part **(d)**, $y(t_{max}) = A(t_{max})$.

The geometric interpretation of the average function $A(t)$ is that it is the height where, to the left of t, the area of the region below the graph of y and above the height is equal to the area above the graph of y and below the height. So, to find the maximum of $A(t)$ we just have to find the point $(t, y(t))$ where the height $y(t)$ has this property. That is, if we look at the horizontal line of height $y(t)$, the area to the left of t, below the graph of y, and above the horizontal line through $(t, y(t))$ is equal to the area to the left of t, above the graph of y, and below the horizontal line. This occurs at approximately $\mathbf{t} = \mathbf{9}$*, where the height is a little less than 50 (actual height is 49). See graphs below.*



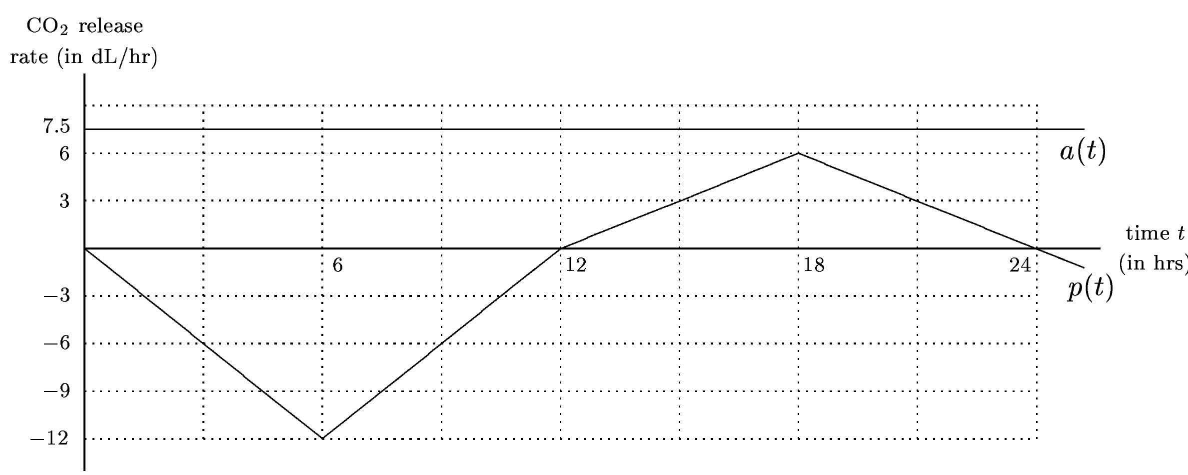
8. (14 points) A team of biologists seeking to develop alternate solutions to the use of pesticides proposes the following experiment. A plant infested by a large colony of aphids (small insects), is placed in a container originally saturated in dioxygen (O_2). They hope that the parasites can be suffocated by the CO_2 produced by their natural activities (e.g. breathing) and those of the plant (e.g. photosynthesis). Make the simplifying assumptions, which are approximations, that:

(i) The container is sealed; no molecules enter or leave.

(ii) The sum of the volumes of O_2 and CO_2 in the container is constant and equal to 117 deciliters (dL); i.e. 11.7 liters.

(iii) At the start of the experiment ($t = 0$), there is no CO_2 in the container.

The light in the room is adjusted so as to mimick a "perfect" 24-hour period. The parasites produce CO_2 at a constant rate of 7.5 dL per hour. When the light is turned on, the plant absorbs CO_2 and releases O_2. When the light is turned off, it does the opposite; i.e. uses O_2 and produces CO_2. The rates of release in the container of CO_2 (in dL per hour) for the parasites, $a(t)$, and the plant, $p(t)$, are plotted below. Note that negative rates correspond to absorption.



(a) By approximately how much does the volume $V(t)$ of CO_2 in the container change in the small time interval between t and $t + \Delta t$? Express your answer in terms of Δt, and of the rates $p(t)$ and $a(t)$.

The production of CO_2 in the tank comes from both the aphids and the plant. So the total rate at which CO_2 is released is given by $a(t) + p(t)$. Thus, in the small time interval from t to $t + \Delta t$, the (small) change in the volume of CO_2 is :

$$\mathbf{\Delta V = \big(a(t) + p(t)\big)\,\Delta t}\,.$$

Solution continued on next page.

Solution continued from previous page.

(b) Write an integral that gives the volume $V(12)$ of CO_2 in the container after 12 hours. Explain why your integral gives the value of $V(12)$. You will probably want to use the answer from **(a)** in your explanation.

We are told that there is no CO_2 in the tank at time $t = 0$, so $V(0) = 0$. To find the volume of CO_2 present in the tank after 12 hours, i.e. $V(12)$, we can "slice" the interval from $t = 0$ to $t = 12$ into small time intervals of length Δt. According to part **(a)**, *during each of these small time intervals, the change in the volume of carbon dioxyde is $\Delta V = \big(a(t) + p(t)\big)\Delta t$. Summing all those little contributions gives a Riemann sum. Letting Δt go to zero, the Riemman sum gives the integral:*

$$\mathbf{V(12) = \int_0^{12} \big(a(t) + p(t)\big)\, dt}\,.$$

(c) What is the value of $V(12)$? Explain how you obtained your answer.

The integral in **(b)** *represents the sum of the area under the graph of $a(t)$ and the area under the graph of $p(t)$, between $t = 0$ and $t = 12$. To find its value, it suffices to count the "boxes" on the chart. The scale on the figure shows that each box represents a volume of 9 dL of CO_2. Counting the boxes gives:*

$$\mathbf{V(12) = (10 - 8) \times 9 = 18} \ \textit{\textbf{deciliters.}}$$

(d) When the container becomes saturated in CO_2, i.e. no O_2 remains, the parasites suffocate to death, and the experiment is stopped. Decide whether that will happen within the 24-hour period; and, if you think it will, estimate this time.

We are told there are 117 dL of O_2 available at $t = 0$. All aphids will thus die if the the volume of CO_2 in the container reaches the value 117. To decide whether that will happen before the end of the experiment, i.e. before $t = 24$ hours, we evaluate $V(24)$ by counting the boxes as was done in **(c)**. *Doing so gives:*

$$V(24) = (20 - 8 + 4) \times 9 = 144 > 117 \ \textit{deciliters.}$$

Accordingly, the aphids **will** *indeed suffocate to death before the experiment ends.*
Again counting the boxes, we find that will happen after **21 hours**. *Indeed, at $t = 21$ the total volume of CO_2 in the container is:*

$$\mathbf{V(21) = (17.5 - 8 + 3.5) \times 9 = 117} \ \textit{\textbf{deciliters.}}$$