MATH 116 — SECOND MIDTERM EXAM
Solutions

Fall 2004

NAME: ______________________________ ID NUMBER: ____________________________

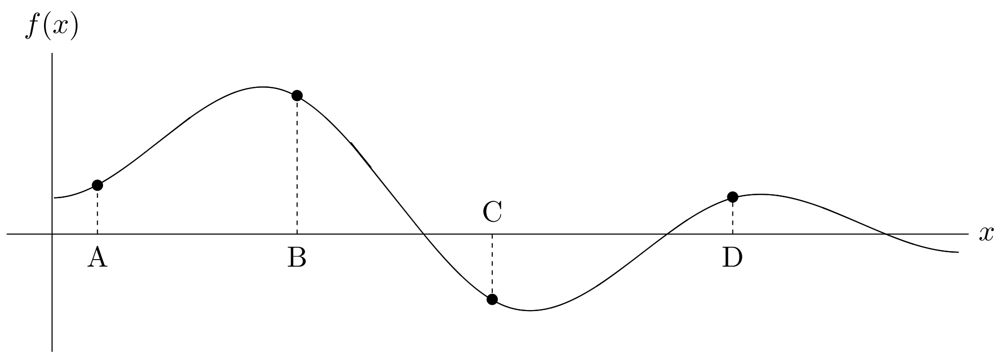
INSTRUCTOR: _____________________ SECTION NO: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and devices whose sounds might disturb your classmates. Please remove **all** headphones.

PROBLEM	POINTS	SCORE
1	16	
2	10	
3	10	
4	18	
5	10	
6	12	
7	11	
8	13	
TOTAL	100	

1. (16 points) For each of the three questions, fill in the blank(s) using the appropriate suggested answer(s). *No explanation is required.*

(a) The polynomial $P_2(x) = 1 + 3(x - a) - 2(x - a)^2$ is the second degree Taylor polynomial approximating the function f for x near a. The graph of f is given in the figure. Which of the points A, B, C, or D on the x-axis has a as its x-coordinate?



ANSWER : **D** .

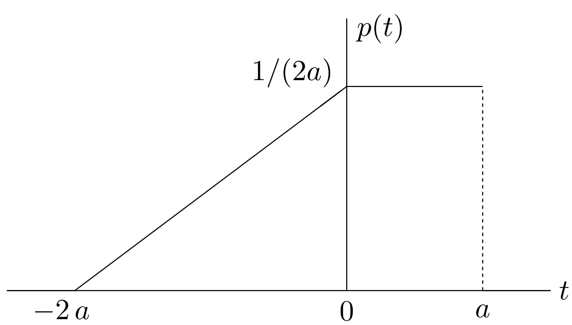
(b) Three of the tests for deciding the convergence or divergence of an infinite series are:

A. integral test, **B.** comparison test, **C.** ratio test.

Using each of these letters **A, B, C** *exactly once*, fill in the blank by each of the following infinite series with the label of the most appropriate test to use in deciding whether the series converges or diverges.

$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ **C** $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1 + n^{5/2}}$ **B** $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ **A**

(c) The graph of the distribution $p(t)$ is shown on the figure, where $a > 0$ is a constant. Fill in the blank with "*greater than*", "*equal to*", or "*smaller than*" to make the sentence below the graph correct.



The median of the distribution p(t) is greater than **its mean.**

Solution continued on next page.

Solution continued from previous page.

(a)

From the Taylor polynomial $P_2(x)$, we deduce the following information:

$$f(a) \;=\; 1\,, \qquad f'(a) \;=\; 3\,, \qquad f''(a) \;=\; -4\,.$$

Thus, near the point $x = a$, the function $f(x)$ is positive, increasing and concave down. From the given graph, the only point which exhibits these features is **D**.

(b)

Because of the "factorial" in the first series, the most appropriate test for deciding whether it converges or diverges is the **ratio test**. *Actually, this series converges.*

For the second series, the most appropriate choice is the **comparison test**. *Indeed, since $\sin^2 n$ is always smaller than 1, while $1 + n^{5/2}$ is always greater than $n^{5/2}$, we find:*

$$\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1 + n^{5/2}} \;<\; \sum_{n=1}^{\infty} \frac{n}{n^{5/2}} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\,,$$

which is well-known to converge. Therefore so does our series, by the comparison test.

For the last series, one uses the **integral test**. *The given series "behaves" like the improper integral $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x}$. By performing the substitution $u = \ln x$, one discovers this integral is divergent, and therefore so is the series. The details are left to the reader.*

(c)

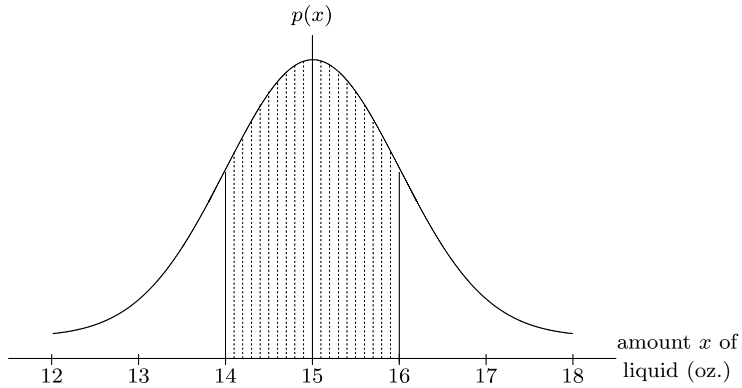
The median of the distribution is clearly zero since the area of the rectangle on the right side of the figure is equal to the area of the triangle on the left. However, the mean must be less than zero because the half of the area that is on the left side of the figure is spread further to the left than the area (the rectangle) on the right, thus making the integral $\int_{-\infty}^{+\infty} tp(t)\,dt = \int_{-2a}^{a} tp(t)\,dt < 0$.

An analytic way to see this geometric fact is to observe that if $-a < t < 0$ and if s is the point symmetric to t with respect to the line $t = -a$, i.e. $(t+s)/2 = -a$ or $s = t - 2a$, then $p(t) + p(s) =$ height of rectangle $=$ constant. Also, $s < t$ so that $(sp(s) + tp(t)) < t(p(t) + p(s)) = t \cdot$ height. Then the mean of the distribution is:

$$\int_{-2a}^{-a} sp(s)\,ds \,+\, \int_{-a}^{0} tp(t)\,dt \,+\, \int_{0}^{a} t\cdot(\mathit{height})\,dt \;=\; \int_{-a}^{0} (tp(t) + sp(s))\,dt + \int_0^a t\cdot(\mathit{height})\,dt$$
$$<\; \int_{-a}^{0} t\cdot(\mathit{height})\,dt + \int_0^a t\cdot(\mathit{height})\,dt \;=\; 0\,.$$

One can also calculate the mean directly and find it is equal to $-a/12$, which is negative and thus indeed smaller than the median.

2. (10 points) A firm that manufactures and bottles apple juice has a machine that automatically fills bottles with 15 ounces (oz) of apple juice. There is some variation, however, in the amount of liquid dispensed in each bottle. Over a long period of time, the average amount dispensed into the bottles was 15 ounces, but the underlying measurements showed the distribution of the ounces, x, of juice in the bottles was given by $p(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-15)^2}$.



(a) What fraction of the bottles contained between 14 and 16 oz of juice? *Explain.*

The fraction of the bottles which contained between 14 and 16 oz of juice is given by:

$$\frac{1}{\sqrt{2\pi}} \int_{14}^{16} e^{-\frac{1}{2}(x-15)^2}\, dx \;\approx\; \mathbf{0.683}\,,$$

using the calculator.
Thus roughly **68%** *of the bottles contained between 14 and 16 oz of juice.*

(b) Give a graphical interpretation of your answer to part **(a)** on the figure.

The integral evaluated in part **(a)** *is simply an expression for the value of the area located between the x-axis, the curve of* $p(x)$*, and the lines* $x = 14$ *and* $x = 16$*. See the figure above.*

(c) Find, as accurately as you can, the fraction of the bottles that contained at least 17 oz of juice inside them. *Explain.*

Similarly to what was done in part **(a)***, the fraction of bottles which contained at least 17 oz of juice is given by the improper integral:*

$$\frac{1}{\sqrt{2\pi}} \int_{17}^{\infty} e^{-\frac{1}{2}(x-15)^2}\, dx\,.$$

We have learned in class this integral converges.
Because $p(x)$ *is a distribution function, the total area under its graph between* $-\infty$ *and* $+\infty$ *must be equal to 1. By symmetry, we conclude the area under the graph of* $p(x)$ *between* $x = 15$ *and* $+\infty$ *is 1/2. Therefore*

$$\frac{1}{\sqrt{2\pi}} \int_{17}^{\infty} e^{-\frac{1}{2}(x-15)^2}\, dx \;=\; \frac{1}{2} \;-\; \frac{1}{\sqrt{2\pi}} \int_{15}^{17} e^{-\frac{1}{2}(x-15)^2}\, dx \;\approx\; \frac{1}{2} - 0.47725 \;\approx\; 0.0277\,.$$

Thus, the fraction of bottles containing at least 17 oz of juice is approximately **0.0227***, i.e. roughly* **2.3%***.*

3. (10 points)

(a) Find the radius of convergence R of the following power series. *Show your work.*

$$\sum_{n=1}^{\infty} \frac{(n + n^3\, 2^n)}{n^2\, 3^n}\, (x-1)^n\,.$$

The general coefficient of the given power series is $a_n = \dfrac{n + n^3\, 2^n}{n^2\, 3^n}(x-1)^n$. *We need to find the limit as* n *goes to infinity of the ratio* $\left|a_{n+1}/a_n\right|$. *For this, observe that for large values of* n, *the term* a_n *"behaves" like* $\dfrac{n^3\, 2^n}{n^2\, 3^n}\,(x-1)^n = n\,(2/3)^n (x-1)^n$. *Thus we have:*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)(2/3)^{n+1}\,|x-1|^{n+1}}{n(2/3)^n\,|x-1|^n} = \frac{2}{3}\lim_{n\to\infty}\frac{n+1}{n}\,|x-1| = \frac{2}{3}\,|x-1|\,.$$

Therefore, the radius of convergence of the power series is $\mathbf{\frac{3}{2}}$.

(b) What is the interval of convergence of the series?

The power series is given around the point $x = 1$, *and we have found its radius of convergence to be* $3/2$. *Accordingly, the series converges for values of* x *within the point* $1 - 3/2 = -1/2$, *and the point* $1 + 3/2 = 5/2$.

The interval of convergence is therefore: $-\mathbf{\frac{1}{2}} < \mathbf{x} < \mathbf{\frac{5}{2}}\,.$

4. (18 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**. *You need not explain your answer.*

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

T [**F**]

(b) The Taylor series for $f(x)g(x)$ near $x = 0$ is $f(0)g(0) + f'(0)g'(0)x + \frac{1}{2}f''(0)g''(0)x^2 + \cdots$.

T [**F**]

(c) If $p(x)$ is the probability density function of some characteristic x distributed throughout a population, then $p(3) = 0.4$ means that 40% of the population has $x < 3$.

T [**F**]

(d) The infinite series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ converges.

T [**F**]

(e) If the Taylor polynomial of degree four of $f(x)$ about $x = 0$ is $2 - 3x + 5x^3 + 7x^4$, then the Taylor polynomial of $g(x) = \frac{f(x^2) - 2}{x}$ of degree five about $x = 0$ is $5x^5 - 3x$.

[**T**] **F**

(f) Let $P(x)$ be the cumulative distribution function of the blood cholesterol level of men aged 40 and over in the US population, measured in milligrams (mg) per deciliter (dL). The equality $P(190) = 0.5$ means that the median blood cholesterol level of men in this population is 190 mg/dL.

[**T**] **F**

5. (10 points)

(a) Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0. *You must show the calculations that lead to your answer.*

We will need two derivatives of $f(x)$. It's not hard to compute $f'(x) = \frac{1}{2}(x+4)^{-1/2}$, and $f''(x) = -\frac{1}{4}(x+4)^{-3/2}$. Hence $f'(0) = 1/4$ and $f''(0) = -1/32$. Since in addition $f(0) = 2$, we obtain the second-order Taylor polynomial for $f(x)$ near $x = 0$:

$$P_2(x) \;=\; f(0) \,+\, f'(0)\,x \,+\, \frac{f''(0)}{2}\,x^2 \;=\; \mathbf{2} + \frac{\mathbf{x}}{\mathbf{4}} - \frac{\mathbf{x^2}}{\mathbf{64}}\,.$$

(b) What is the Taylor series about $x = 0$ of the function $\sin x$? *No explanation required.*

This series is well-known. Near $x = 0$, we have

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,.$$

(c) Using your answers to parts **(a)** and **(b)** and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4+\sin(2x)}$ for x near 0. *Show your work.*

Of course, there's no need to differentiate $g(x)$ to derive the polynomial. Indeed, observe that $g(x) = f\big(sin(2x)\big)$; so we can make use of our answers to parts **(a)** *and* **(b)**.

From part **(b)**, *we see that $\sin(2x) \approx 2x$ for x near 0. Thus, for small values of x, the function $\sin(2x)$ has values near zero. Thus, from part* **(a)**, *we deduce:*

$$g(x) \;\approx\; P_2\big(\sin(2x)\big) \;=\; 2 + \frac{\sin(2x)}{4} - \frac{\sin^2(2x)}{64}\,.$$

Next, from part **(b)**, *we get:*

$$\sin(2x) \;\approx\; 2x \;-\; \frac{(2x)^3}{3!}\,.$$

Substituting this in the approximation we found for $g(x)$ gives us:

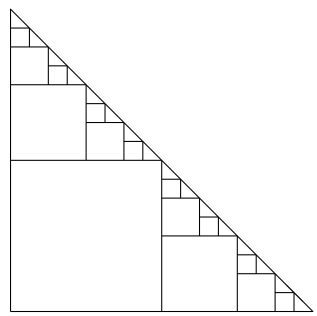
$$g(x) \;\approx\; 2 + \frac{2x - 8x^3/3!}{4} - \frac{\big(2x - 8x^3/3!\big)^2}{64}\,.$$

To finish the work, there remains to truncate the latter to second order. Hence we obtain:

$$g(x) \;\approx\; 2 + \frac{2x}{4} - \frac{(2x)^2}{64} \;=\; \mathbf{2} + \frac{\mathbf{x}}{\mathbf{2}} - \frac{\mathbf{x^2}}{\mathbf{16}}\,,$$

which is the desired second-order Taylor polynomial near $x = 0$ for the function $g(x)$.

6. (12 points) We have learned how to use slicing to calculate areas and volumes. This problem explores a different kind of slicing through a simple example. A right-isoceles triangle with sides of length 2 is covered by squares as illustrated and explained in the figure below.



step 1: one square of side length 1

step 2: two squares of side length 1/2

step 3: four squares of side length 1/4

step 4: eight squares of side length 1/8

... etc ...

(a) Use a geometric series to find the area covered by the squares after the N^{th} step.

After the 1st step, the area covered is $1 \cdot (1)^2$.

After the 2nd step, the area covered is $1 \cdot (1)^2 + 2 \cdot (1/2)^2$.

After the 3rd step, the area covered is $1 \cdot (1)^2 + 2 \cdot (1/2)^2 + 4 \cdot (1/4)^2$.

After the 4th step, the area covered is $1 \cdot (1)^2 + 2 \cdot (1/2)^2 + 4 \cdot (1/4)^2 + 8 \cdot (1/8)^2$.

Continuing this pattern, we find that the area covered after the N^{th} *step is given by* $\displaystyle\sum_{j=0}^{N-1} 2^j \left(\frac{1}{2^j}\right)^2$.

We may simplify this expression and use the formula for the sum of a finite geometric series so as to obtain:

$$\textit{Area covered after N steps} = \sum_{j=0}^{N-1} \left(\frac{1}{2}\right)^j = \frac{1-(1/2)^N}{1-1/2} = \mathbf{2}\left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2^N}}\right).$$

(b) Use your answer to part **(a)** and your knowledge of series to find the total area covered by the infinitely many squares.

All there is to do is to let N go to infinity in the formula we found in the previous question. Hence we find:

$$\textit{Total area covered} = \lim_{N\to\infty} 2\left(1 - \frac{1}{2^N}\right) = 2\,.$$

Therefore, after infinitely many steps, the total area covered is **2**.

(c) How do you know your answer to part **(b)** is the correct one?

We know it is correct because it is equal to the area of the original right-isoceles triangle of side length 2, as expected.

7. (11 points) Suppose that as you finish your degree at Michigan, you develop a great idea for starting a company that, you believe, is dead certain to start out making a profit at a rate of \$40,000 per year (after your salary and all expenses). Further, your idea is so good that the profits will increase in future years so that t years after starting the company, it will be making profits at a rate of $40,000 + 10,000t$ dollars per year. Assuming your projections are correct,

(a) How much profit will your company make in the first 10 years of operation?

The profit made in the period of time from t to $t + \Delta t$ is approximately $(40,000 + 10,000t)\Delta t$. Summing this up over all times in the first ten years, we get a Riemann sum whose limit as $\Delta t \to 0$, is the integral of $40,000 + 10,000t$ from 0 to 10. Thus, the total profit earned in these years is

$$\int_0^{10} (40,000+10,000t)\,dt = 400,000+10,000\int_0^{10} t\,dt = 400,000+500,000 = \mathbf{900,000} \;\; \textit{\textbf{dollars.}}$$

(b) What is the value at the time your company is started of the first 10 years of its profits? Assume an interest rate of 8% per year, compounded continuously. Be sure to both write an integral whose value is equal to the present value you are computing and evaluate the integral.

The present value at startup of the company of the profits earned in the time period between t and $t + \Delta t$ years is approximately $e^{-0.08t}(40,000 + 10,000t)\Delta t$ dollars, since the interest rate is 8% compounded continuously. As in part **(a)***, we again add these up over times in the first 10 years and let $\Delta t \to 0$ to get the present value of*

$$\begin{aligned}\int_0^{10} e^{-0.08t}(40,000 + 10,000t)\,dt &= 40,000\left(\frac{1 - e^{-0.8}}{0.08}\right) + 10,000\left((1-t)\frac{e^{-0.08t}}{0.08}\bigg|_0^{10}\right) \\ &\approx 275,325.52 + 298,762.29 = \mathbf{574,097.81} \quad \textit{\textbf{dollars.}}\end{aligned}$$

(c) Being a bright and inquisitive person, you'd like to sell your company after a few years to pursue other interests. One common way of assigning a value to a company is that it is worth, "the present value of all its future earnings". Assuming your profit projections are correct, the same interest rate of 8% with continuous compounding as in part **(b)**, and receiving this value for your company when you sell it in 10 years, write an integral whose value is equal to the amount you would receive at that time.

As in part **(a)***, the amount of profits in the time period from t to $t + \Delta t$ is approximately $(40,000 + 10,000t)\Delta t$ dollars, which has a value of $e^{-0.08(t-10)}(40,000 + 10,000t)\Delta t$ dollars ten years after startup, the time the company is to be sold. We again add up for $10 \le t \le T$, take the limit as $\Delta t \to 0$, and then the limit as $T \to \infty$ to see that the value of the company is*

$$\int_{10}^{\infty} (40,000 + 10,000t)\, e^{-.08(t-10)}\, dt \quad \textit{dollars.}$$

The numerical value of the integral is 3,313,000, so the company has a value at that time of about **\$3.3 million***.*

8. (13 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e \;, \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the third order Taylor polynomial for the function $\sqrt{\dfrac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2}$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the Doppler Law for slow-moving emitters:

$$F_r \approx \left(1 - \frac{v}{c}\right) F_e \,.$$

If we substitute $-v/c$, which we are told is very small, for x in the given Taylor polynomial, we obtain:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e = \left(1 - \frac{v}{c} + \frac{(-v/c)^2}{2} + \frac{(-v/c)^3}{2}\right) F_e + \cdots$$

$$= \left(1 - \frac{v}{c}\right) F_e + \frac{v^2}{2c^2} F_e - \frac{v^3}{2c^3} F_e + \cdots \,.$$

Truncating the latter gives the desired approximation for slow-moving emitters.

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 10% of the speed of light c? *Explain.*

The error made when approximating the Doppler Law by the relationship given in part **(a)** *is the sum of infinitely many powers of v/c. We found above the first two of these terms are* $\dfrac{v^2}{2c^2} F_e$ *and* $\dfrac{-v^3}{2c^3} F_e$.

Accordingly, we may use $\dfrac{v^2}{2c^2} F_e$ *as a good approximation for the error.*

If the velocity is at most 10% of the speed of light, then $v/c \le 0.1$. Hence we deduce

$$\textit{Approximate Error} \le \frac{1}{2}(0.1)^2 F_e = 0.005\, F_e \,.$$

Thus we find the "error" to be less than half of 1% of F_e, which is indeed within the suggested bound.