MATH 116 — FINAL EXAM

Solutions

Fall 2004

NAME: ____________________________ ID NUMBER: ____________________________

INSTRUCTOR: ____________________ SECTION NO: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 12 pages including this cover. There are 11 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about the problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and devices whose sounds might disturb your classmates. Please remove **all** headphones.

PROBLEM	POINTS	SCORE
1	16	
2	6	
3	5	
4	6	
5	10	
6	15	
7	5	
8	6	
9	10	
10	8	
11	13	
TOTAL	100	

1. (16 points)

(a) If $\int_0^1 f(x)\,dx = 2$, then $\int_0^2 f\left(\frac{x}{2}\right)\,dx =$ $\underline{\quad \mathbf{4} \quad}$.

By substituting $u = x/2$, we obtain $dx = 2\,du$, and the interval of integration $0 \le x \le 2$ changes to $0 \le u \le 1$. Hence

$$\int_0^2 f\left(\frac{x}{2}\right)\,dx \;=\; 2\int_0^1 f(u)\,du \;=\; 2\cdot 2 \;=\; \mathbf{4}\,.$$

(b) Does the infinite series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge? *Justify your answer.*

We will use the Ratio Test with the general term $a_n = \frac{2^n}{n!}$. Doing so yields:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \;=\; \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\,\frac{n!}{(n+1)!} \;=\; 2\lim_{n\to\infty}\frac{1}{n+1} \;=\; 0\,,$$

which is smaller than 1. Therefore, the Ratio Test allows us to conclude that the given series **converges**.

Another solution consists in noting that since $e^x = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!}$ holds true for all values of x, the given series is actually **equal to** $e^2 - 1$, *which is finite. Therefore, the series converges.*

(c) Is the function te^{-2t} a solution of the differential equation $\frac{dy}{dt} + 2y - e^{-2t} = 0$? *Explain why or why not.*

If $y = te^{-2t}$, then

$$\frac{dy}{dt} \;=\; e^{-2t} - 2\,te^{-2t} \;=\; e^{-2t} - 2y\,.$$

Therefore, indeed, the given function does satisfy the differential equation, **it is a solution**.

(d) Suppose $C(t)$ is the daily cost of heating your house, measured in dollars per day, where $t = 0$ corresponds to January 1, 2004. Give the meaning, in words, of each of the following quantities.

(i) $\int_0^{60} C(t)\,dt\,.$

This integral represents the **total cost in dollars** *of heating your house during the first 60 days of the year 2004, i.e. throughout January and February 2004.*

(ii) $\frac{1}{60}\int_0^{60} C(t)\,dt\,.$

This quantity represents the **average cost in dollars per day** *of heating your house during the first 60 days of the year 2004, i.e. throughout January and February 2004.*

2. (6 points) Write a parametrization for each of the following curves in the xy-plane.

(a) The circle of radius 2, centered at the origin, traced clockwise, and starting from $(-2, 0)$ when $t = 0$.

One possible parametrization (there are many others) is:

$$x(t) = -2\cos(t), \quad y(t) = 2\sin(t), \quad 0 \le t < 2\pi.$$

(b) The line passing through the points $(2, -1)$ and $(1, 3)$.

One possible parametrization (there are many others) is:

$$x(t) = t, \quad y(t) = -4t + 7.$$

3. (5 points)

(a) Briefly explain the difference between the indefinite integral $\int f(x)\,dx$ and the (proper) definite integral $\int_a^b f(x)\,dx$.

(See the text, §6.2, third paragraph on page 268.) "The definite integral is a number and the indefinite integral $\int f(x)dx$ is a family of functions."

The indefinite integral is the family of all functions that are antiderivatives of f. Any two functions in the family differ by a constant. The definite (proper) integral $\int_a^b f(x)dx$ is a number. When $f(x) \ge 0$, it is equal to the area of the region above the x-axis and below the graph of the function $f(x)$ between $x = a$ and $x = b$.

(b) What is a Riemann sum and how is it related to one or more of the integrals in part **(a)**?

(See the text, §5.2, box on page 234.) A Riemann sum for f on an interval $[a, b]$ is a sum of the form

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

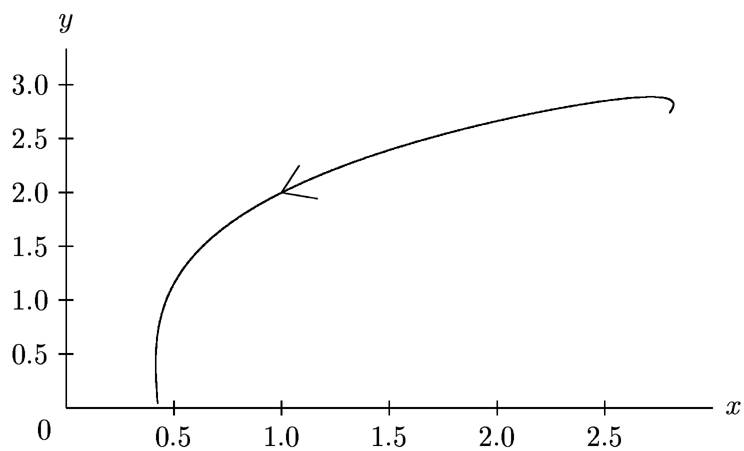
where $a = x_0 < x_1 < \cdots < x_n = b$, and, for $i = 1, 2, \ldots, n$, $\Delta x_i = x_i - x_{i-1}$, and $x_{i-1} \le c_i \le x_i$. The definite integral $\int_a^b f(x)\,dx$ is defined to be the limit of the Riemann sums when the maximum length of the intervals $[x_{i-1}, x_i]$ tends to 0. Thus, a Riemann sum can be viewed as an approximation to the definite integral.

Geometrically, the Riemann sum may be viewed as the sum of areas of rectangles of height $f(c_i)$ and width $x_i - x_{i-1}$, a geometric approximation to the area of the region under the graph of f.

4. (6 points) A particle moves in the xy-plane so that it is at the position $(x(t), y(t))$ at time t, where $x(t)$ and $y(t)$ satisfy the system of differential equations

$$\frac{dx}{dt} = x^2 - y^2\,, \qquad \frac{dy}{dt} = x - 2t\,.$$

It is known that at time $t = 2$, the particle is at the point $(1, 2)$. A graph of the path of the particle is shown in the figure.



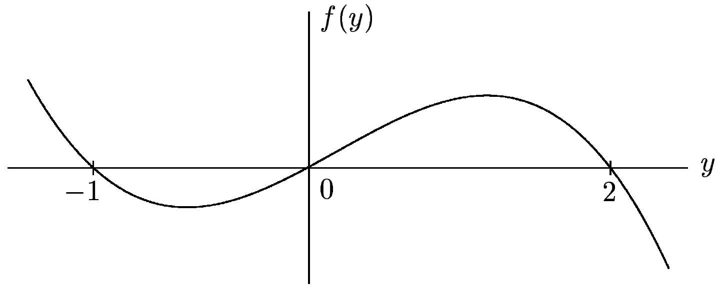
Find the instantaneous velocity of the particle at time $t = 2$, and draw an arrow along the curve that shows the direction of motion. *Show your work.*

Substituting $t = 2$, $x = 1$, and $y = 2$ in the given differential equations, we find that when $t = 2$, we have $dx/dt = -3$ and $dy/dt = -3$. Thus at $t = 2$, the particle's instantaneous velocity is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2} \simeq 4.24\,.$$

Moreover, we can see that the x-position and y-position of the particle are both decreasing (as their derivatives with respect to time are negative). Therefore, at the point $(1, 2)$, **the particle follows the path "downhill"**, *as shown in the figure.*

5. (10 points) Let f be the function whose graph is given on the figure below.



(a) On page 6, six field plots are displayed. Choose the one that corresponds to the differential equation $\dfrac{dy}{dx} = f(y)$.

From the graph of $f(y)$, we see that the differential equation has three equilibrium solutions, namely at $y = -1$, $y = 0$, and $y = 2$. Thus we may automatically rule out Plots A, B and E which do not have horizontal lines at these values of y.

When $y < -1$, the value of $f(y)$ is positive so the slope field must have positive slopes in this range. Since the slopes in Plots C and D are negative in this range, they are not possible.

Therefore, it must be Plot F.

ANSWER : **F**.

(b) Find all the equilibrium solutions of the differential equation $\dfrac{dy}{dx} = f(y)$.

The constant or equilibrium solutions are $y(x) = -1$, $y(x) = 0$, and $y(x) = 2$, since $f(y)$ vanishes at these values, as was already mentioned in part **(a)**.

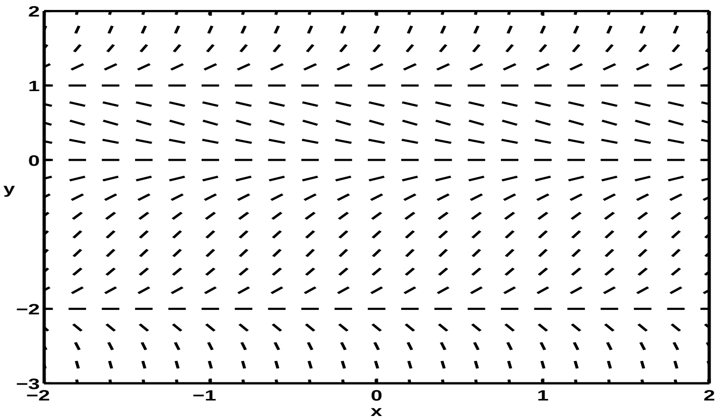
(c) Which of the equilibrium solutions you found in part **(b)** are stable? *Explain the reason(s) for your answer(s).*

The solution $\mathbf{y(x) = -1}$ **is stable**. *Because, if $y(x)$ is a solution so that y is slightly smaller than -1, the function $f(y)$ is positive, so that $dy/dx = f(y(x))$ is positive. Accordingly, the solution $y(x)$ is increasing when y is below the equilibrium $y = -1$. This means that solutions increase toward $y = -1$ from below. Also, when y is slightly larger than -1, the function $f(y)$ is negative, which shows that dy/dx is negative. Accordingly, the solutions decrease toward $y = -1$ above and near the equilibrium $y = -1$.*

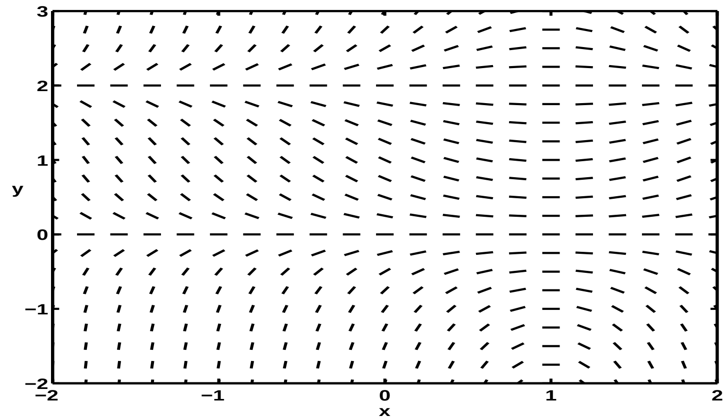
A similar argument also applies to the equilibrium solution $\mathbf{y = 2}$ **which is stable**. *The important properties are that $f > 0$ for y near 2 but $y < 2$ and $f < 0$ for y near 2 but $y > 2$.*

The equilibrium solution $\mathbf{y = 0}$ **is unstable**, *because $f(y) < 0$ for y near 0 but $y < 0$ and $f(y) > 0$ for y near 0 but $y > 0$. Solutions $y(x)$ that are near 0 decrease and move farther from 0 if $y < 0$ and solutions near 0 but with $y > 0$ are increasing so also move farther away from 0.*

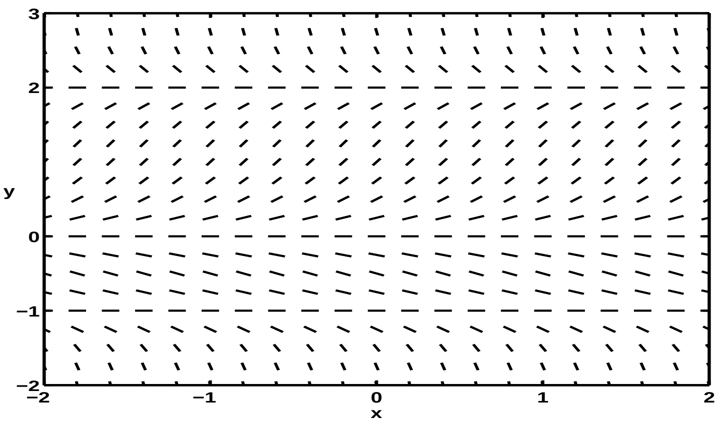
The plots shown on this page refer to Problem 5 on page 5.



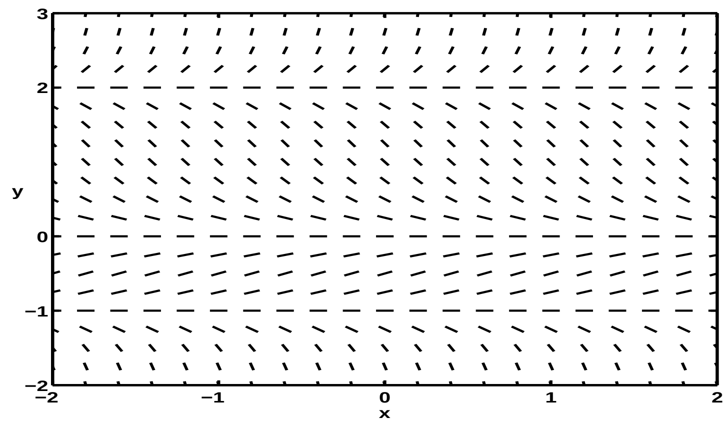
Plot A



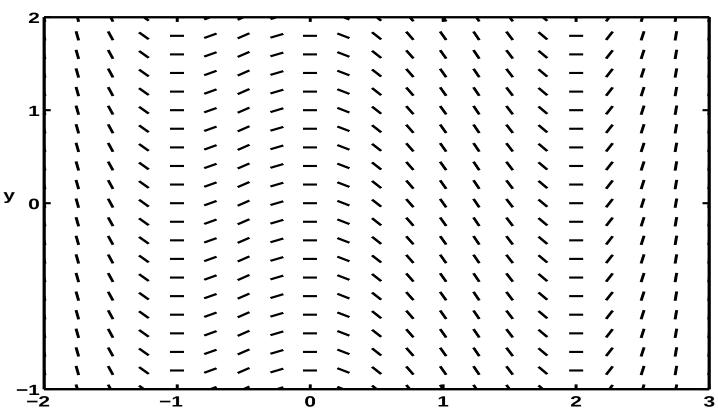
Plot B



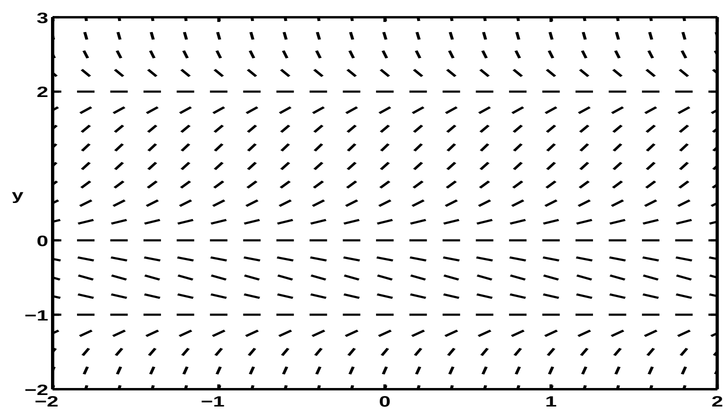
Plot C



Plot D



Plot E



Plot F

6. (15 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**. *You need not explain your answer.*

(a) The formula $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ holds for all real numbers $x \neq 1$ and all positive integers $n = 1, 2, 3, \ldots$.

[**T**] **F**

(b) If $g(x)$ is a periodic function, then every solution $y = f(x)$ of the differential equation $\dfrac{dy}{dx} = g(x)$ is also a periodic function.

T [**F**]

(c) If $y = f(t)$ is a solution of the differential equation $\dfrac{dy}{dt} = y^2 - t$, then for every constant C, $f(t) + C$ is also a solution of the differential equation.

T [**F**]

(d) The function $y(t) = 0$ is a solution of the initial value problem

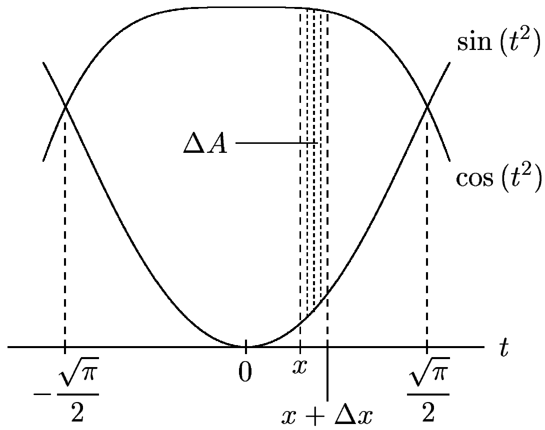
$$\frac{dy}{dt} = 3t - y^3\,, \quad y(0) = 0.$$

T [**F**]

(e) There is a solution of the logistic differential equation $\dfrac{dP}{dt} = 0.03P\left(1 - \dfrac{P}{3}\right)$ that satisfies $P(1) = 1$ and $P(20) = 5$.

T [**F**]

7. (5 points) For $-\frac{\sqrt{\pi}}{2} \leq x \leq \frac{\sqrt{\pi}}{2}$, let $A(x)$ be the area of the region bounded by the curves $\cos(t^2)$, $\sin(t^2)$, and the vertical lines $t = -\frac{\sqrt{\pi}}{2}$ and $t = x$. See the figure below.



(a) Sketch on the figure an area that represents $\Delta A = A(x + \Delta x) - A(x)$ for a small number Δx.

See the picture above.

(b) Find a formula for the derivative $A'(x)$.

When Δx is very small, the area ΔA is well approximated by the area of the box of width Δx and height $\cos(x^2) - \sin(x^2)$. Hence we find $\Delta A \simeq \left[\cos(x^2) - \sin(x^2)\right] \Delta x$.

Thus, we have $\frac{\Delta A}{\Delta x} \simeq \cos(x^2) - \sin(x^2)$. Letting Δx go to zero, the ratio $\frac{\Delta A}{\Delta x}$ approaches $A'(x)$, which is how we arrive at the answer given below.

ANSWER : $\mathbf{A'(x)} = \underline{\cos(\mathbf{x^2}) - \sin(\mathbf{x^2})}$.

8. (6 points) For what values of the positive number p does the infinite series $\sum_{n=1}^{\infty} \frac{n^3 - 4n^2}{n^p + 5}$ converge? *Explain the reason for your answer.*

As n gets very large, the numerator of the general term of the given series "behaves" like n^3, while its denominator "behaves" like n^p. Thus, our series is comparable to the series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^{p-3}}.$$

From classwork, we know such a series converges if and only if $p - 3 > 1$, i.e. if and only if $p > 4$.

ANSWER : $\underline{\mathbf{p > 4}}$.

9. (10 points) At age 65, Mrs. Smith retires with $1,000,000 in her retirement account. Assume that after retirement:

(i) She receives interest of 5% per year (compounded continuously) on the balance in the account, and this money is reinvested in the account ;

(ii) She withdraws money (for living expenses) from the account at a continuous rate of $60,000 per year.

(a) Write the initial value problem for the balance $B(t)$ of dollars in the account t years after Mrs. Smith retires.

From the information given in the problem, we deduce that the balance in dollars, $B(t)$, in Mrs. Smith's account satisfies the differential equation

$$B'(t) \;=\; 0.05\,B(t) \;-\; 60,000$$

and has the initial value $B(0) = 1,000,000$ dollars at time 0.

(b) Will Mrs. Smith ever exhaust the retirement account, i.e. reduce the balance in the account to zero? *Explain.*

The amount of money B in the account is being reduced at all times. Indeed, originally $dB/dt = 0.05B - 60,000 < 50,000 - 60,000 < 0$. As B decreases, so does $0.05B$. Hence this also happens at all future times. In particular, the account decreases by at least $10,000 each year, and it will thus eventually be depleted.

Although the problem does not require it, one can find out when the account will be depleted by solving the equation $B(T) = 0$, for T. The solution to the differential equation in part **(a)** *is $B(t) = 1,200,000 + A\,e^{0.05t}$, where A is a constant. Since $B(0) = 1,000,000$, we get $A = -200,000$ and thus $B(t) = 1,200,000 - 200,000\,e^{0.05t}$. The solution of the equation $B(T) = 0$ is found to be $T = (\ln 6)/0.05 \approx 36$ so it will take about 36 years to deplete the account.*

(c) Are there any equilibrium solutions to the differential equation of part **(a)**? If so, explain their meaning in terms of Mrs. Smith's money.

The only equilibrium solution of the differential equation from part **(a)** *is where $dB/dt = 0$ for all times t or where $B = 60,000/0.05 = 1,200,000$.*

This is the amount of money necessary so that the interest generated is exactly equal to the amount that Mrs. Smith is withdrawing, so that the balance in the account remains constant.

10. (8 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e\,, \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the Taylor series for the function $\sqrt{\frac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \cdots$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the approximation to the Doppler Law for slow-moving emitters:

$$F_r \simeq \left(1 - \frac{v}{c}\right) F_e\,.$$

If we substitute $-v/c$, which we are told is very small, for x in the given Taylor series, we obtain:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e = \left(1 - \frac{v}{c} + \frac{(-v/c)^2}{2} + \frac{(-v/c)^3}{2}\right) F_e + \cdots$$

$$= \left(1 - \frac{v}{c}\right) F_e + \frac{v^2}{2c^2} F_e - \frac{v^3}{2c^3} F_e + \cdots\,.$$

Truncating the latter gives the desired approximation for slow-moving emitters.

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 20% of the speed of light c? *Explain.*

The error made when approximating the Doppler Law by the relationship given in part **(a)** *is the sum of infinitely many powers of v/c multiplied by F_e. We found above the first two of these terms are $\frac{v^2}{2c^2} F_e$ and $\frac{-v^3}{2c^3} F_e$.*

Accordingly, we may use $\frac{v^2}{2c^2} F_e$ as a good approximation for the error.

If the velocity is at most 20% of the speed of light, then $v/c \leq 0.2$. Hence we deduce

$$\textit{Approximate Error} \leq \frac{1}{2}(0.2)^2 F_e = 0.02\, F_e = 2\,\%\, F_e\,.$$

Thus we do not find the "error" to be less than 1% of F_e. Therefore, one should not believe that the error made falls within the suggested bound.

11. (13 points) In normal conditions, the thyroid hormone (Hormone T), produced in the thyroid gland, and the thyroid successor hormone (Hormone S), produced in the pituitary gland, form a so-called "auto-regulated feedback process". The amount of one in the bloodstream influences the production of the other, and vice-versa. The simple system given below models this process, where x is the amount of Hormone T (in standard units), and y is the amount of Hormone S (in standard units), present in the bloodstream at time t hours.

$$\frac{dx}{dt} = 3 - y\,, \qquad \frac{dy}{dt} = x - 2\,.$$

(a) Find all equilibrium solutions (if any) of the system.

To find the equilibrium solutions of a system of differential equations, we must solve the equations $dx/dt = 0$ and $dy/dt = 0$ simultaneously for x and y. For our system, doing so immediately shows that there is only one equilibrium solution, namely $\mathbf{x(t) = 2}$ and $\mathbf{y(t) = 3}$.

(b) Suppose that at $t = 0$, the amount of Hormone T in the blood was 1.0 and the amount of Hormone S was 3.5, both in standard units. Find the equation of the trajectory of the corresponding solution curve in the phase plane. *Show your work.*

We first convert the system of two equations into a single differential equation linking y and x together without the variable t. That equation is:

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{x-2}{3-y}\,.$$

The given initial conditions becomes the single condition $y(1.0) = 3.5$.

Separating the variables produces

$$(3-y)\,dy = (x-2)\,dx\,.$$

We integrate both sides of this last equation to discover the relationship

$$-\frac{1}{2}(3-y)^2 + C = \frac{1}{2}(x-2)^2\,,$$

where C is an arbitrary integration constant whose value may be determined precisely by using the condition $y(1.0) = 3.5$. Doing so yields

$$-\frac{0.5^2}{2} + C = \frac{(-1)^2}{2} \quad\Longrightarrow\quad C = \frac{5}{8}\,.$$

Therefore, the equation of the trajectory with the given initial conditions is

$$\mathbf{(y-3)^2 + (x-2)^2 = \frac{5}{4}}\,.$$

You may recognize this equation as that of the circle centered at the point $(2,3)$ of radius $\sqrt{5}/2$. One should not be a surprised to find the trajectory to be a circle. Indeed, we are told that in normal conditions, the hormonal pair is auto-regulated. This means that the amount of each hormone in the bloodstream is periodic in time. We will see in the second part of this problem that when the system is deregulated, the solution curve is no longer a circle.

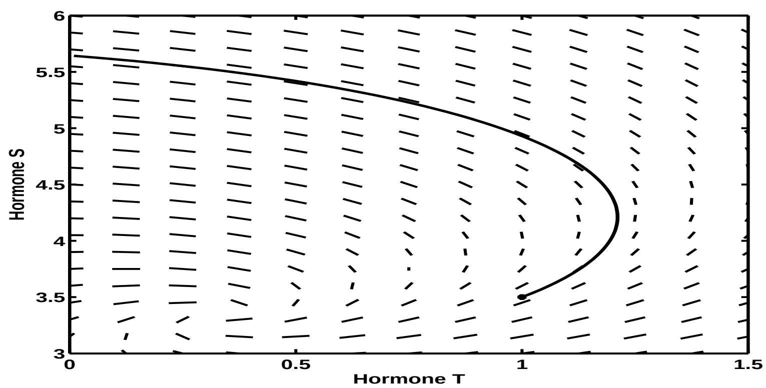
Solution continued on next page.

Solution continued from previous page.

If the patient's diet lacks iodine (e.g. from salt), the chemical agent responsible for detecting the presence of Hormone S in the blood is no longer active. The above model must be replaced by the new system:

$$\frac{dx}{dt} = 3 - y + x\,, \qquad \frac{dy}{dt} = x - 2 + \frac{y}{2}\,.$$

The slope field for the differential equation that describes the trajectories of this system is shown on the figure below.



(c) Sketch on the figure the trajectory corresponding to the initial values in part **(b)**; that is, $x(0) = 1.0$ and $y(0) = 3.5$. *You need not solve any differential equation.*

To sketch the trajectory, we first locate the initial point which we are told has coordinates $(1.0, 3.5)$. *Then we have to decide which direction the curve follows. Substituting* $x = 1.0$ *and* $y = 3.5$ *in the new system, we find* $dx/dt = 1.5$ *and* $dy/dt = 0.75$, *which are both positive. Accordingly, the trajectory moves in the positive* x *and positive* y *directions. Following the slope field in that direction from the initial point, we arrive at the curve shown above.*

(d) In the context of this problem, briefly describe how the amounts of the hormones change from their initial values as time increases.

For a short while, roughly up to the point $(1.2, 4.2)$, *both hormones increase in amount. Probably, the hormonal system is not totally deregulated yet. Things turn bad past the point* $(1.2, 4.2)$. *The amount of Hormone T continually decreases, while the amount of Hormone S continually increases. This pattern continues until the trajectory hits the* y*-axis, roughly at the point* $(0.0, 5.6)$.

The hormonal system is totally deregulated. The presence of Hormone S is no longer detected, and the body fills this apparent lack by producing more and more of it. In turn, Hormone S inhibits the production of Hormone T whose amount decreases until none is left in the blood.

The model given here, although greatly simplified, depicts an actual medical condition known as "goiter": the thyroid gland inflates due the large quantity of Hormone S stagnating in it.