Math 116 – First Midterm Exam

SOLUTIONS

Name: ______________________________

Instructor: ____________________ Section Number: ____________

1. **Do not open this exam until you are told to begin.**
2. This exam has 11 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you obtained your answer.
8. Please turn **off** all cell phones and pagers and **remove** all headphones.

Problem	Points	Score
1	8	
2	13	
3	11	
4	12	
5	11	
6	13	
7	12	
8	10	
9	10	
Total	100	

1. (8 points) Consider the functions f and g defined below. Assume a is a nonzero constant.

$$f(x) = \frac{(x-a)^2}{\sqrt{x}}\ , \qquad g(x) = x\cos(ax),$$

(a) (4 pts.) Find the family of antiderivatives of $f(x)$. Show step-by-step work.

We have

$$f(x) = \frac{x^2 - 2ax + a^2}{\sqrt{x}} = x^{3/2} - 2ax^{1/2} + a^2x^{-1/2}.$$

So,

$$\int f(x)\ dx = \frac{2}{5}x^{5/2} - \frac{4}{3}ax^{3/2} + 2a^2x^{1/2} + C.$$

(b) (4 pts.) Find the family of antiderivatives of $g(x)$. Show step-by-step work.

Using integration by parts,

$$u = x \qquad\qquad u' = 1$$

$$v' = \cos(ax) \qquad\qquad v = \frac{\sin(ax)}{a}.$$

Then,

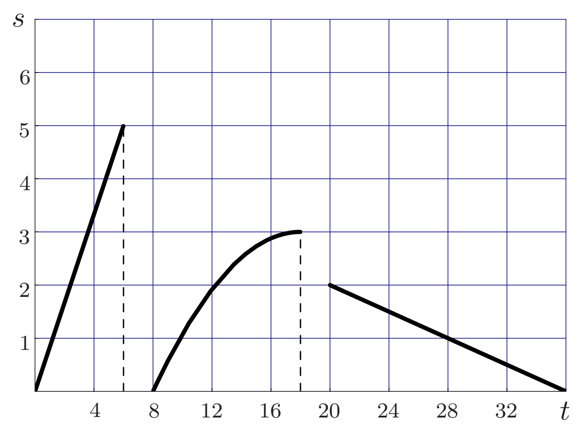
$$\begin{aligned}\int x\cos(ax)\ dx =& \frac{x}{a}\sin(ax) - \int \frac{\sin(ax)}{a}\ dx \\ =& \frac{x}{a}\sin(ax) + \frac{\cos(ax)}{a^2} + C.\end{aligned}$$

2. (13 points) In the magical tale of Harry Potter and the Half-blood Prince, Harry and professor Dumbledore go in search of a horcrux, a dark magic device created by the dark wizard Voldemort in order to hide and preserve a piece of his soul.

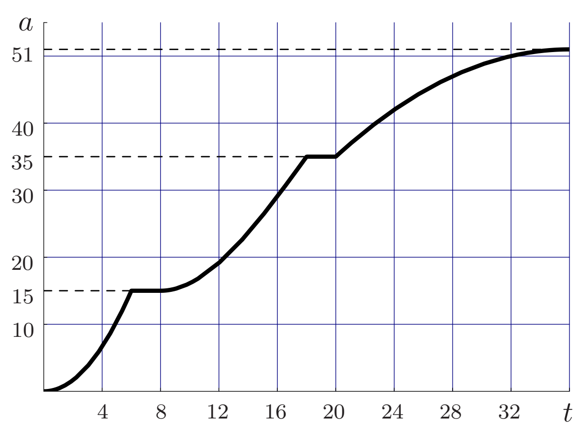
To get to the horcrux, Dumbledore must drink a lethal green elixir filled with dark power. Harry's task is to scoop some of the magic liquid into a goblet, feed it to Dumbledore, then turn around and scoop some more liquid into the goblet, repeating these steps until the elixir is gone.

Harry notices that, after the first glass, Dumbledore's drinking speed increases at a decreasing rate. By the end, Dumbledore is drinking as slowly as possible.

The graph below represents Dubledore's drinking speed, s (in fluid ounces per second) against time, t (in seconds.)



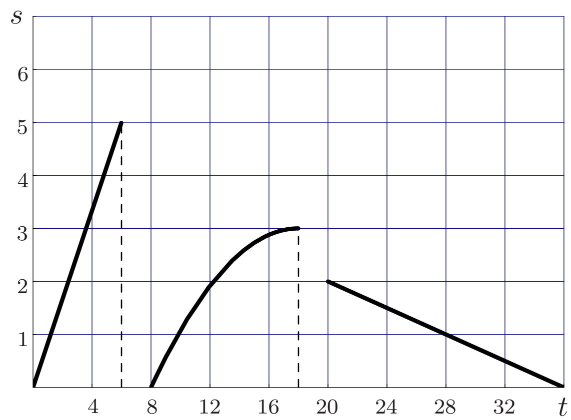
(a) (5 pts.) Sketch the amount a of the elixir that Dumbledore has drunk as a function of time t during the first 32 seconds after he started drinking. Include scales on each axis, and label them appropriately. Clearly indicate on your sketch the heights of the graph of $a(t)$ at $t = 6$, $t = 18$ and $t = 36$.



(This problem continues on the next page.)

(This is a continuation of Problem 2. For your convenience, the original graph is reprinted here.)

The graph below represents Dubledore's drinking speed, s (in fluid ounces per second) against time, t (in seconds).



(b) (4 pts.) Write an expression that represents Dumbledore's average drinking speed during the first 12 seconds after he starts drinking. Use the graph to estimate this value. *Show step-by-step work.*

$$\begin{aligned}\frac{1}{12}\int_0^{12} s(t)\ dt &= \frac{1}{12}\left[\int_0^{6} s(t)\ dt + \int_8^{12} s(t)\ dt\right]\\ &\simeq \frac{1}{12}[15+4]\\ &= \frac{19}{12}\ \text{fluid oz./second}\end{aligned}$$

(c) (4 pts.) *Exactly* how long does it take Dumbledore to drink the last 4 fluid ounces of the dark-magic substance? *Briefly explain your work.*

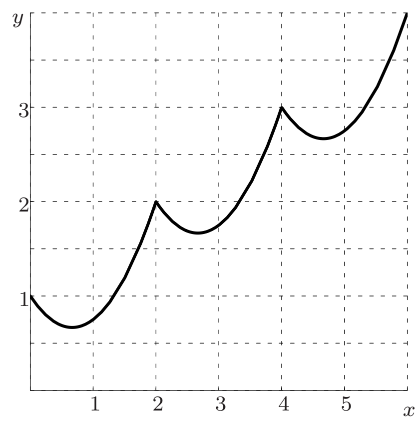
We are looking for a number b such that

$$\int_{36-b}^{36} s(t)\ dt = 4.$$

The last section of the graph is linear, so the area underneath the graph is represented by a triangle. We can see from the graph that if $b = 8$, then the area from $t = 28$ to $t = 36$ is (1/2) 8=4.

It follows that $b = 8$ (seconds) or, in words, it takes Dumbledore exactly 8 seconds to drink the last 4 oz. of liquid.

3. (11 points) The following is a graph of the function $f(x)$ on the interval $[0, 6]$.



(a) (2 pts.) Use the graph to estimate $\int_0^6 f(x)\,dx$ using LEFT(3). (Show your work.)

$$\text{LEFT}(3) = 2\,[f(0) + f(2) + f(4)] = 2\,[1 + 2 + 3] = 12.$$

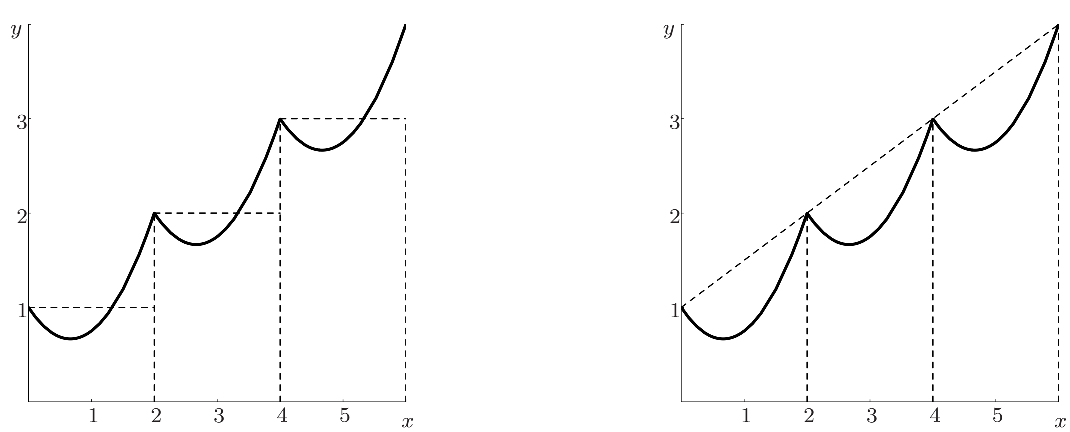
(b) (2 pts.) Use the graph to estimate $\int_0^6 f(x)\,dx$ using TRAP(3). (Show your work.)

$$\text{RIGHT}(3) = 2\,[f(2) + f(4) + f(6)] = 2\,[2 + 3 + 4] = 18.$$

So,

$$\text{TRAP}(3) = \frac{12 + 18}{2} = 15.$$

(c) (4 pts.) The graph given above is reproduced twice for you below. On these graphs, sketch the areas given by the approximations LEFT(3) and TRAP(3). Use the left graph for LEFT(3) and the right graph for TRAP(3).



(d) (3 pts.) Which is a better estimate of $\int_0^6 f(x)\,dx$, LEFT(3) or TRAP(3)? Explain.

LEFT(3) is better. TRAP(3) is an overestimate, because its trapezoids are always above the graph. LEFT(3) cuts through the graph and so balances its over– and underestimates.

4. (12 points) Suppose that $h(x)$ is a continuous function. Suppose also that $H(x)$ is an antiderivative of $h(x)$.

   (a) (3 pts.) What can you say about the relationship between $H(x)$ and $\int_a^x h(t)\,dt$?

   $H(x)$ and $\int_a^x h(t)\,dt$ are both antiderivatives of $h(x)$, so they differ by a constant.

   (b) (3 pts.) Calculate $\dfrac{d}{dx}\displaystyle\int_a^x h(t)\,dt$.

   $$h(x)$$

   (c) (3 pts.) Calculate $\dfrac{d}{dx}\left(\displaystyle\int_a^b h(x)\,dx\right)$.

   $$0$$

   (a definite integral is a constant.)

   (d) (3 pts.) Calculate $\displaystyle\int_a^b \left(\frac{d}{dx}h(x)\right)\,dx$.

   $$h(b) - h(a)$$

5. (11 points) A child is sitting on a Ferris wheel. If the origin is at the center of the circle and we measure x and y in meters, her motion is given by the following parametric equations:

$$x = 125\sin((2\pi/9)t), \qquad y = -125\cos((2\pi/9)t),$$

where we measure t in minutes since she boarded the ride.

(a) (2 pts.) What is the diameter of the Ferris wheel?

$$D = 250 \text{ meters}$$

(b) (2 pts.) How long does it take for the Ferris wheel to make one complete revolution?

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9 minutes

(c) (3 pts.) Find the speed of the child 10 minutes into the ride.

$$\frac{dx}{dt} = \frac{2\pi}{9}125 \ \cos\left(\frac{2\pi}{9}t\right), \qquad \frac{dy}{dt} = \frac{2\pi}{9}125 \ \sin\left(\frac{2\pi}{9}t\right)$$

$$S = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{2\pi}{9}\ 125\right)^2} \simeq 87.26 \text{ meters/minute}$$

(d) (4 pts.) If at 10 minutes into the ride the child were to suddenly step off of the Ferris wheel, her motion would initially be along the tangent line at $t = 10$. Determine parametric equations for this tangent line.

We have,

$$x(10) = 125\sin\left(\frac{2\pi}{9}\ 10\right) \simeq 80$$
$$y(10) = -125\cos\left(\frac{2\pi}{9}\ 10\right) \simeq -95.$$

Also,

$$\left.\frac{dx}{dt}\right|_{t=10} = \frac{2\pi}{9}\ 125\cos\left(\frac{2\pi}{9}\ 10\right) \simeq 67$$
$$\left.\frac{dy}{dt}\right|_{t=10} = \frac{2\pi}{9}\ 125\sin\left(\frac{2\pi}{9}\ 10\right) \simeq 56.$$

So,

$$x(t) = 67\ t + 80$$
$$y(t) = 56\ t - 95.$$

6. (13 points) Let a and b be positive numbers. The region bounded by the positive y-axis, the positive x-axis, the vertical line $x = b$ and the curve $y = e^{-ax}$ is revolved about the x-axis.

(a) (6 pts.) Find the volume of the resulting solid. (Yes, your answer will involve a and b.)

Have,

$$\text{Volume of a slice } (\text{V}_{\text{slice}}) \simeq \pi r^2 \Delta x, \qquad \text{where } r = e^{-ax}.$$

Therefore,

$$\text{V}_{\text{slice}} \simeq \pi (e^{-ax})^2 \Delta x = \pi e^{-2ax} \Delta x.$$

and so

$$\begin{aligned} \text{Total Volume } &= \int_0^b \pi e^{-2ax}\, dx = \left. \frac{\pi e^{-2ax}}{-2a} \right|_0^b \\ &= \frac{\pi e^{-2ab}}{-2a} - \frac{\pi}{-2a} \\ &= \frac{\pi}{2a}(1 - e^{-2ab}). \end{aligned}$$

(b) (7 pts.) Suppose we let $b \to \infty$, creating a solid with an infinitely long neck. Does this solid have finite volume? If so, find it (showing step-by-step work.) If not, explain why not.

$$V = \lim_{b \to \infty} \frac{\pi}{2a}(1 - e^{-2ab}) = \frac{\pi}{2a}\ ,$$

$$\text{since } e^{-2ab} \to 0 \text{ as } b \to \infty, (\text{because } a > 0).$$

So,

$$V = \frac{\pi}{2a} \text{ is finite.}$$

7. (3 points each) Multiple choice. Circle *the single* correct answer to each one of the following questions. (No partial credit will be awarded.)

(I) Suppose that f and g are positive, continuous functions on the interval $x \geq 1$ such that $f(x) \geq g(x)$ for $1 \leq x < 4$ and $f(x) \leq g(x)$ for $x \geq 4$. If $\int_4^\infty g(x)\,dx$ converges to the value A, then what can you say about $\int_1^\infty f(x)\,dx$?

(a) It may diverge or converge; it cannot be determined.

(b) It converges to a number that must be greater than A.

(c) It converges to a number that must be less than A.

$\boxed{\text{(d)}}$ It converges to a number that may be greater than A or may be less than A.

(II) Suppose that $\int_3^\infty \frac{100}{x^p}\,dx$ diverges for some $p \neq 1$. What can you conclude about $\int_0^3 \frac{100}{x^p}\,dx$?

(a) It also diverges.

$\boxed{\text{(b)}}$ It converges.

(c) It may diverge or converge; it cannot be determined.

(d) It is not an improper integral, so we don't talk about its convergence or divergence.

(III) Which of the following is true about $\displaystyle\int_2^\infty \frac{1}{\sqrt{\theta}+\theta^2}\,d\theta$?

$\boxed{\text{(a)}}$ It converges, by comparison with $\int_2^\infty 1/\theta^2\,d\theta$.

(b) It diverges, by comparison with $\int_2^\infty 1/\sqrt{\theta}\,d\theta$.

(c) It diverges, by comparison with $\int_2^\infty 1/\theta^2\,d\theta$.

(d) It converges, by comparison with $\int_2^\infty 1/\sqrt{\theta}\,d\theta$.

(IV) Suppose that $\int_1^\infty A(t)\,dt$ converges, and $A(t)$ is a positive, continuous function for all real numbers. Then, what can you say about the convergence or divergence of

$$\int_1^\infty (A(t)+A(t))\,dt \quad \text{and} \quad \int_1^\infty \left(A(t)+\frac{1}{t}\right)\,dt?$$

$\boxed{\text{(a)}}$ $\int_1^\infty (A(t)+A(t))\,dt$ converges, but $\int_1^\infty (A(t)-+1/t)\,dt$ diverges.

(b) $\int_1^\infty (A(t)+A(t))\,dt$ diverges, but $\int_1^\infty (A(t)+1/t)\,dt$ converges.

(c) They both diverge.

(d) They both converge.

8. (10 points) Circle City is a circular city with a radius of five miles. A straight highway runs East-West through the center of the city. The density of the population at a distance y (in miles North or South) from the road is well approximated by

$$D(y) = 4y$$

(in thousands of people per square mile.) Apparently many people in Circle City like to live as far from the highway as possible.

(a) (4 pts.) Write a Riemann sum that approximates the total population of Circle City.

Since the density depends only on y, it is nearly constant on horizontal slices. Slicing horizontally, we have,

$$\text{Area of a slice} \simeq 2r\Delta y, \qquad \text{where } r = \sqrt{25 - y^2},$$

and so,

$$\text{Total Population of a slice} \simeq 8y\sqrt{25 - y^2}\ \Delta y.$$

Summing up, we have

$$2\sum 8y\sqrt{25 - y^2}\ \Delta y,$$

where the factor "2" is due to the symmetry of areas and density with respect to the highway.

(b) (4 pts.) Write an integral that gives the total population of Circle City.

$$2\int_0^5 8y\sqrt{25 - y^2}\ dy$$

(c) (2 pts.) Evaluate your integral to find the total population of Circle City.

Using integration by substitution, or a calculator, we see that the total population of Circle City is

$$2\int_0^5 8y\sqrt{25 - y^2}\ dy = \frac{16}{3}(25)^{3/2} \simeq 667 \text{ thousand people.}$$

9. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) The integral $\int \arcsin(x)\ dx$ can be integrated by parts.

$\boxed{\text{TRUE}}$ FALSE

(b) The graph of the equation $r = \theta$ is a straight line.

TRUE $\boxed{\text{FALSE}}$

(c) The integral

$$\int_0^{\pi/2} \frac{1}{2}(5\sin 2\theta^2)\ d\theta,$$

represents the area enclosed by one petal of the rose curve $r = 5\sin 2\theta$.

$\boxed{\text{TRUE}}$ FALSE

(d) The area of a circular oil spill grows at a rate of $r(t)$ square miles per hour, where t is measured in hours. Then $\int_0^3 r(t)\ dt$ gives the total change (in miles) in the radius of the spill during the first three hours after it occurred.

TRUE $\boxed{\text{FALSE}}$

(e) The integral $\int_0^r \pi(r^2 - y^2)\ dy$ represents the total volume of a sphere of radius r.

TRUE $\boxed{\text{FALSE}}$