Math 116 – Second Midterm Exam

SOLUTIONS

Name: ______________________________

Instructor: ______________________ Section Number: ______________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	14	
4	8	
5	14	
6	10	
7	10	
8	9	
9	11	
Total	100	

1. (12 points) Let $f(x) = 2e^{x/2}$.

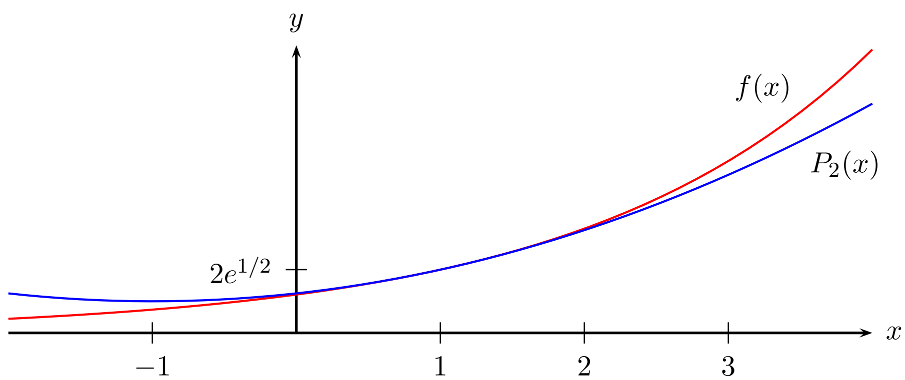
(a) (4 pts.) Find $P_2(x)$, the Taylor polynomial for $f(x)$ of degree 2 centered at $x = 1$.

$$
\begin{array}{rcl|rcl|rcll}
f(x) & = & 2e^{x/2} & f(1) & = & 2e^{1/2} & f(1)/0! & = & 2e^{1/2} & \approx 3.2974 \\
f'(x) & = & e^{x/2} & f'(1) & = & e^{1/2} & f'(1)/1! & = & e^{1/2} & \approx 1.6487 \\
f''(x) & = & \frac{1}{2}e^{x/2} & f''(1) & = & \frac{1}{2}e^{1/2} & f''(1)/2! & = & \frac{1}{4}e^{1/2} & \approx 0.4122
\end{array}
$$

So

$$
\begin{aligned}
P_2(x) &= e^{1/2}\left(2 + (x-1) + \frac{1}{4}(x-1)^2\right) \\
&\approx 3.2974 + 1.6487(x-1) + 0.4122(x-1)^2.
\end{aligned}
$$

(b) (3 pts.) Graph the functions $f(x)$ and $P_2(x)$ for $0 \le x \le 2$ on the same set of axes. Label each function clearly.



(c) (2 pts.) Use the polynomial $P_2(x)$ that you wrote in part (a) to estimate $f(0.1)$ and $f(1.1)$.

$$P_2(0.1) = 2.1475 \qquad P_2(1.1) = 3.4664$$

(d) (3 pts.) Briefly demonstrate which of the previous two approximations is more accurate.

Compare the approximations with the actual values of $f(x)$:

x	$P_2(x)$	$f(x)$
0.1	2.1475	2.1025
1.1	3.4664	3.4665

Clearly $P_2(1.1)$ is closer to $f(1.1)$ than $P_2(0.1)$ is to $f(0.1)$. That is, the approximation at $x = 1.1$ is more accurate than the approximation at $x = 0.1$. That's as expected: the approximation should be better nearer to $x = 1$, since the polynomial is expanded about that point.

2. (12 points) Problems (a) and (b) below are independent of each other.

(a) (6 pts.) Consider the following statement: "If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges." Is the statement true or false?

TRUE $\boxed{\text{FALSE}}$

If you said "true," give a step-by-step argument that shows the statement is always true. If you said "false," then write down a specific series for which the statement is false (you must give an explicit formula).

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{(Note: } 1/n \to 0 \text{ as } n \to \infty, \text{ yet this series diverges by the integral test.)}$$

(b) (6 pts.) Consider the following statement: "The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges." Is the statement true or false?

$\boxed{\text{TRUE}}$ FALSE

Give a step-by-step argument to justify your answer.

This is an alternating series. Since

$$0 < 1/(n+1) < 1/n, \qquad \text{and} \qquad \lim_{n\to\infty} (1/n) = 0,$$

the given series converges by the alternating series test.

3. (14 points) Please note that the two parts of this problem involve different power series.

(a) (8 pts.) Use the ratio test to find the *radius of convergence*, R, for the series

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

Show step-by-step work.

We use the ratio test. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}}{(n+1)!}x^{n+1}}{\frac{3^n}{n!}x^n}\right| = |x| \; \frac{3^{n+1}}{(n+1)!} \; \frac{n!}{3^n} = |x| \frac{3}{n+1}.$$

Since

$$\lim_{n\to\infty} |x| \frac{3}{n+1} = |x| \lim_{n\to\infty} \frac{3}{n+1} = 0,$$

the radius of convergence is $R = \infty$. That is, this series converges for all x in $(-\infty, \infty)$.

(b) (6 pts.) The power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n^3},$$

has a radius of convergence $R = 2$ (so we know that this series converges at least on the open interval $(-1, 3)$.) Find the *interval of convergence* for this series. Show step-by-step work.

To find the interval of convergence we test the endpoints of the open interval of convergence.

- Test $x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3-1)^n}{2^n n^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}$,
  which converges by the alternating series test (or by the fact that the series of absolute values, namely $\sum 1/n^3$ converges by the integral test.)

- Test $x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-1-1)^n}{2^n n^3} = \sum_{n=0}^{\infty} \frac{1}{n^3}$, which converges by the integral test.

So the interval of convergence is $[-1, 3]$.

NOTE: observe that in its current form, this series is undefined at $n = 0$. If you noticed so explicitly in the exam and solved the problem accordingly, you received the appropriate credit. The solution just described is correct if the series started at, $n = 1$ for instance.

4. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) The sum of the finite geometric series $\sum_{n=0}^{81} ax^n = a + ax + ax^2 + \cdots + ax^{81}$ is $\dfrac{a(1-x^{81})}{1-x}$ provided that $x \neq 1$.

TRUE $\boxed{\text{FALSE}}$

(b) Let $F(x)$ be the cumulative distribution function of the heights of grass plants in a meadow, measured in meters. The statement $F(0.5) = 0.25$ means that 25% of the grass plants in the meadow have a height of at most 0.5 meters.

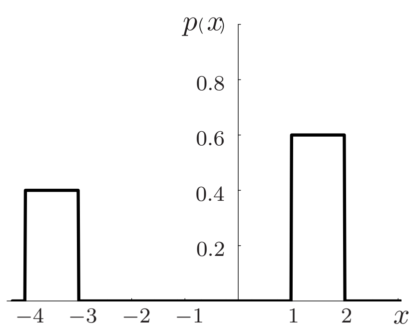
$\boxed{\text{TRUE}}$ FALSE

(c) Let $f(x)$ be the probability density function of the heights of grass plants in a meadow, measured in meters. The statement $f(0.5) = 0.7$ means that 70% of the grass plants in the meadow have height very close to 0.5 meters.

TRUE $\boxed{\text{FALSE}}$

(d) Let a quantity have density function $p(x)$, where the graph of $p(x)$ is shown in the figure. The median of the quantity is positive.

$\boxed{\text{TRUE}}$ FALSE



5. (14 points) Congratulations! You have just won the latest jackpot of the Eterna-Lottery. This lottery offers its winners payments in perpetuity (lasting forever) that can be transfered to the winner's decedents. For your prize you are offered your choice of one of the following two payment plans.

   (a) Lump sum payments of \$7,500,000 on November 16 every year starting now and lasting forever

   (b) A continuous income stream of \$8,000,000 starting now and lasting forever

   Assuming that the interest rate is, and always will remain, 8 percent compounded continuously, find the present value of both of the payment plans. Which plan would you choose based on consideration of present values? Clearly and carefully show all of your work.

(a) Present Value (in millions of dollars) $= 7.5 + 7.5e^{-0.08} + 7.5e^{-2(0.08)} + 7.5e^{-3(0.08)} + \cdots$

Since this is a geometric series with radius $r = e^{-0.08}$, and $-1 < e^{-0.08} < 1$, the series converges. Its sum is given by

$$\sum_{n=0}^{\infty} 7.5(e^{-0.08})^n = \frac{7.5}{1 - e^{-0.08}} \simeq 97.55 \text{ million dollars.}$$

(b) Here we compute the present value of the continuous income stream (assuming the units of the given rate are \$8,000,000 *per year*):

$$\begin{aligned}
\text{Present Value (in millions of dollars)} &= \int_{n=0}^{\infty} 8e^{-0.08t}\, dt \\
&= \lim_{b\to\infty} \int_{n=0}^{b} 8e^{-0.08t}\, dt \\
&= \lim_{b\to\infty} 8/(-0.08)e^{-0.08t}\Big|_{n=0}^{b} \\
&= \lim_{b\to\infty} 8/(-0.08)[e^{-0.08b} - 1]\Big|_{n=0}^{b} \\
&= 8/0.08 \simeq 100 \text{ million dollars.}
\end{aligned}$$

One should choose plan (b), as it has the highest present value!

6. (10 points) A quantity has density function $p(x)$, where

$$p(x) = \begin{cases} 0 & \text{when } x < 0, \\ a + bx^2 & \text{when } 0 \le x \le 1, \\ 0 & \text{when } x > 1. \end{cases}$$

Assuming that the mean value of the quantity is $\frac{3}{4}$, find a and b. Show your work.

We know

$$\begin{aligned} 1 &= \int_0^1 (a + bx^2)\, dx = \left(ax + b\,\frac{x^3}{3} \right)\Bigg|_0^1 = a + \frac{b}{3}, \\ 3/4 &= \int_0^1 (ax + bx^3)\, dx = \left(a\,\frac{x^2}{2} + b\,\frac{x^4}{4} \right)\Bigg|_0^1 = \frac{a}{2} + \frac{b}{4}. \end{aligned}$$

The first equation tells us that $a = 1 - b/3$. Substituting this into the second equation, yields $b/12 = 1/4$ or $b = 3$. Substituting this value into the first equation tells us that $a = 0$.

Thus, $a = 0$ and $b = 3$.

7. (10 points)

(a) (2 pts.) What is the Taylor series about $x = 0$ of the function e^x? *No explanation or work required.*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

(b) (8 pts.) *Without computing any derivatives*, find the first four nonzero terms of the Taylor series for the function $g(x) = e^{\sin x}$. Show step-by-step work.

Since,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we then have:

$$\begin{aligned} e^{\sin x} &= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + \cdots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{3!} + \cdots\right)^4 + \cdots \\ &= 1 + x + \frac{1}{2}x^2 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{24}\right)x^4 + \cdots \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots \end{aligned}$$

8. (3 points each) Multiple choice. Circle *the* correct answer to each one of the following questions. (No partial credit!)

(a) Suppose that the power series $\sum c_n x^n$ converges for $x = -3$ and diverges for $x = 8$. Then, which of the following claims are necessarily true?

(I) Its radius of convergence could be π.
(II) The series must diverge at $x = -8$.
(III) The series must converge at $x = 3$.
(IV) The series must diverge at $x = 9$.

(A) Statements (I), (III) and (IV) only.

$\boxed{\text{(B)}}$ Statements (I) and (IV) only.

(C) Statements (III) and (IV) only.

(D) All statements are true.

(b) Assume $\lim_{n\to\infty} S_n = \sqrt{2(0.1)}/(1-(0.1))$, where $S_1, S_2, \cdots, S_n, \cdots$ is the sequence of partial sums for a geometric series. Then, which of the following claims are necessarily true?

(I) The geometric series just mentioned converges.
(II) The first term of the geometric series just mentioned must be the number 0.1.
(III) The geometric series just mentioned could have the form $\sum_{n=0}^{\infty} \sqrt{2(0.1)}\, 0.1^n$.
(IV) The geometric series just mentioned may diverge or converge; it cannot be determined.

(A) Statements (III) and (IV) only.

(B) Statements (I) and (II) only.

$\boxed{\text{(C)}}$ Statements (I) and (III) only.

(D) Statements (II) and (IV) only.

(c) Consider the following sequences. Assume a and r are positive constants.

(I) $S_n = (-1)^n \cos(n\pi)$, (II) $S_n = ar^n$, (III) $S_n = \dfrac{1}{\ln(5^n) + 1{,}000{,}000}$.

What can you say about the convergence or divergence of each of the above?

(A) They all converge.

(B) Sequence (I) converges, sequence (II) may diverge or converge, and (III) diverges.

$\boxed{\text{(C)}}$ Sequence (I) converges, sequence (II) converges sometimes, and (III) converges to 0.

(D) Sequence (I) diverges, sequence (II) converges as long as $|r| \leq 1$, and (III) converges to 0.

9. (11 points) The theory of relativity predicts that when an object moves at speeds close to the speed of light, the object appears heavier. The apparent, or relativistic, mass m, of the object when it is moving at speed v is given by the formula

$$m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

where c is the speed of light and m_0 is the mass of the object when it is at rest.

(a) (8 points) Write the first four nonzero terms of the Taylor series for m in terms of v. (Hint: You may want to use the binomial series.)

Since m is given by

$$m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

and $0 \leq v < c$, we may use the binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots,$$

with $x = -v^2/c^2$ and $p = -1/2$, to obtain:

$$m = m_0 \left(1 + \frac{1}{2c^2}v^2 + \frac{3}{8c^4}v^4 + \frac{5}{16c^6}v^6 + \cdots\right).$$

(b) (3 points) The series you derived in part (a) converges for v in the interval $[0, c)$. Interpret the practical significance of this interval of convergence in the context of this problem (that is, as far as the relativistic mass of an object is concerned.)

When an object's speed (v) is less than the speed of light (c) the relativistic mass (m) of the object is finite. In this case, the Taylor series for the relativistic mass (derived above) converges to the actual relativistic mass.