Math 116 – Final Exam
solutions

Name: ____________________

Instructor: ____________________ Section Number: ____________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and remove all headphones.

Problem	Points	Score
1	12	
2	10	
3	9	
4	8	
5	12	
6	18	
7	12	
8	9	
9	10	
Total	100	

1. (12 points) The world shrimp production can be represented by the differential equation

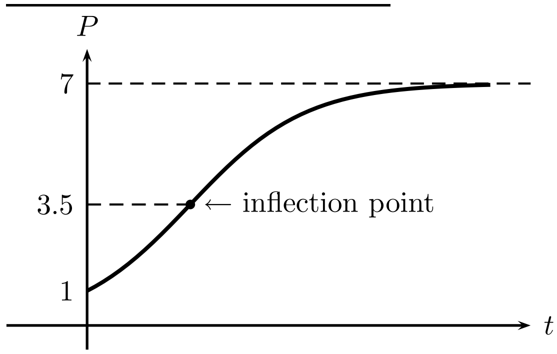
$$\frac{dP}{dt} = -0.1P(P-7),$$

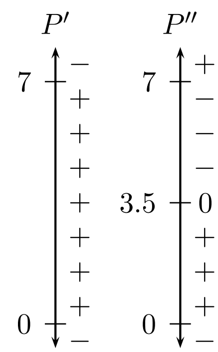
where t is the number of years since 1982 and $P(t)$ is the quantity of shrimp farmed in the world during year t in hundreds of thousands of metric tons. In 1982 the world shrimp production was 100,000 metric tons.

(a) (3 pts.) Determine all of the equilibrium solutions of the differential equation given above. Classify each as either stable or unstable. No explanation required.

- $P = 0$ (unstable equilibrium;)
- $P = 7$ (stable equilibrium.)

(b) (4 pts.) Sketch a graph of the solution to the given initial value problem. Be sure to indicate clearly on your graph where the solution curve is increasing/decreasing and where it is concave up/concave down. Clearly mark the value of any asymptotes.





Note

$$P''(t) = \frac{d}{dt} - 0.1P(P-7)$$
$$= -0.1(2P-7)\frac{dP}{dt}$$
$$= .01P(P-7)(2P-7).$$

(c) (3pts.) Use Euler's method with $\Delta t = 0.5$ to estimate the world shrimp production in the year 1984 ($t = 2$).

t_i	P_i	slope at (t_i, P_i)	$\Delta P_i = \Delta t \times$ (slope at (t_i, P_i))
0.0	1.000	0.600	0.300
0.5	1.300	0.741	0.371
1.0	1.671	0.890	0.445
1.5	2.116	1.033	0.517
2.0	**2.632**		

So, the world's shrimp production in 1984 was approximately 263,200 metric tons.

(d) (2 pts.) Is the estimate of world shrimp production in part (c) bigger or smaller than the exact solution to the initial value problem at $t = 2$? Explain in one sentence.

The estimate is smaller than the actual value. The exact solution curve is concave up when P is between 1 and 2.632, so a tangent line-based approximation to the actual solution curve yields an underestimate to the actual values.

2. (10 points) An apple is placed in a room whose air temperature is fixed at 50 °F. Let $T(t)$ be the temperature of the apple at time t, which is measured in hours. According to Newton's Law of Heating and Cooling, the rate of change of the apple's temperature satisfies

$$\frac{dT}{dt} = k(T - 50).$$

(a) (2 pts.) What is the value of k if the temperature decreases at an instantaneous rate of 3 °F per hour when the temperature T of the apple is 65?

$$\frac{dT}{dt} = -3, \text{ when } T = 65, \quad \text{so}$$

$$-3 = k\ (65 - 50), \quad \text{or } k = -0.2.$$

(b) (5 pts.) Now assume $k = -0.1$. Solve the initial value problem $dT/dt = k(T - 50)$ with $T(0) = 30$.

Separating variables, we have

$$\int \frac{1}{T-50}\ dT = \int k\ dt$$
$$\ln|T - 50| = kt + C$$
$$T = Ae^{kt} + 50; \quad \text{where } A \text{ is an arbitrary constant.}$$

Now, using the initial condition (0,30), we get $30 = A + 50$ or $A = -20$.
So,

$$T = 50 - 20e^{-0.1t}.$$

(c) (3 pts.) Briefly explain what the solution from part (b) says about the temperature of the apple over the time interval $[0, \infty)$.

The apple progressively warms up (initially rather fast, then slower and slower) from its initial temperature of 30° F to just about 50° F in the long run (as this is the temperature of the surrounding air.

3. (9 points) Using the techniques of integration that you have learned in Math 116 (and not calculator integration), integrate and/or evaluate exactly each of the integral expressions below. Show all your work.

   (a) (3 pts.) $\displaystyle\int \frac{\ln(2t)}{t}\,dt$

   Substituting

$$u = \ln(2t), \quad \text{so that} \quad du = \frac{1}{t}dt,$$

   we immediately get

$$\int \frac{\ln(2t)}{t}\,dt = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln(2t))^2 + C.$$

   (b) (6 pts.) $\displaystyle\int_3^\infty xe^{-x}\,dx$

   Integrating by parts, using

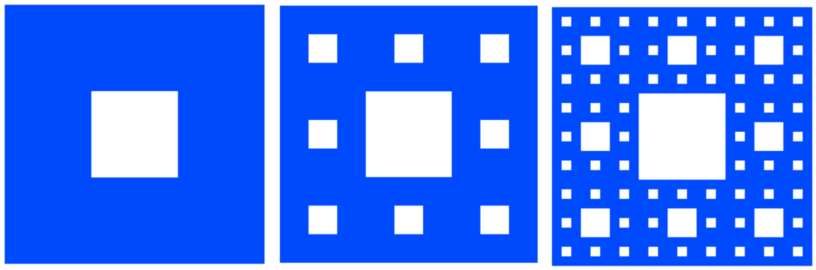
$$\begin{array}{ll} u = x & u' = 1 \\ v' = e^{-x} & v = -e^{-x}, \end{array}$$

   yields

   - $\displaystyle\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -e^{-x}(1+x) + C;$

   - $\displaystyle\lim_{b\to\infty}\int_3^b xe^{-x}\,dx = \lim_{b\to\infty}[-e^{-x}(1+x)]|_3^b = \lim_{b\to\infty}[-e^{-b}(1+b)] - [-e^{-3}(1+3)] = 4e^{-3}$

4. (8 points) The Sierpinski Carpet is an example of a mathematical object called a fractal. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure below shows the first three steps of the construction.)



At the n-th step of the process, 8^{n-1} squares are removed, each with area $\left(\frac{1}{9}\right)^n$. Thus, the area removed at the n-th step is $A_n = \left(\frac{8^{n-1}}{9^n}\right)$. There are infinitely many steps in the process.

(a) (2 pts.) Find the limit of the sequence $A_1, A_2, A_3, \ldots$.

$$\lim_{n\to\infty}\left(\frac{8^{n-1}}{9^n}\right) = \frac{1}{9}\lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0, \qquad \text{as } 8/9 < 1.$$

(b) (2 pts.) Write a mathematical expression that represents A, the *total* sum of the areas of the removed squares after infinitely many steps of the process.

$$A = A_1 + A_2 + A_3 + \cdots = \sum_{n=1}^{\infty}\left(\frac{8^{n-1}}{9^n}\right) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n.$$

(c) (4 pts.) Exactly how much area is removed in all? Show your work.

Since the infinite sum

$$A = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n$$

is a geometric series with ratio $8/9 < 1$, we have:

$$A = (1/9)\,\frac{1}{1-8/9} = (1/9)\ 9 = 1.$$

So, the total area removed from the carpet is equal to 1.

5. (12 points) Determine whether each of the following series converges or diverges. Circle CONVERGES or DIVERGES and then BRIEFLY EXPLAIN why each series converges or diverges. In each part of the problem you will receive one point for circling the correct answer (and only the correct answer) and up to two points for your explanation.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ $\boxed{\text{DIVERGES}}$ CONVERGES

Explanation:

Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$, the terms of the series do not approach zero, which means that their infinite sum diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^5+2}$ DIVERGES $\boxed{\text{CONVERGES}}$

Explanation:

For large n, $n^3/(n^5+2) \simeq n^3/n^5 \simeq 1/n^2$. So the given series behaves like $\sum 1/n^2$, which converges by the integral test.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\boxed{\text{DIVERGES}}$ CONVERGES

Explanation:

Since $\int 1/(x \ln x)\ dx = \int 1/u\ du = \ln|u| + C = \ln|\ln x| + C$, and $lim_{b\to\infty} \ln|\ln b| = \infty$, the sum diverges by the integral test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ DIVERGES $\boxed{\text{CONVERGES}}$

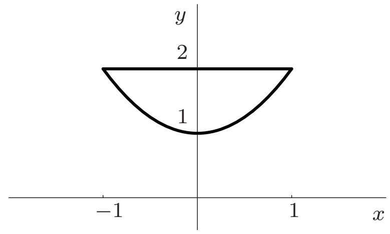
Explanation:

Using the ratio test,

$$\lim_{n\to\infty} \left[\frac{(n+1)^2 2^{n+2}}{3^{n+1}} \frac{3^n}{n^2 2^{n+1}} \right] = \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} = 2/3.$$

Since $2/3 < 1$, we have convergence.

6. (18 points) A thin metal plate lying in the region bounded by the line $y = 2$ and the parabola $y = x^2 + 1$ has uniform density 5 gm/cm^2.



(a) (2 pts.) Write an integral expression giving the exact *area* of this region. *Do not* evaluate this expression.

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$$4 - 2\int_0^1 (1 + x^2)\, dx$$

(b) (4 pts.) Write an integral expression giving the exact *perimeter* of this region. *Do not* evaluate this expression.

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$$2 + 2\int_0^1 \sqrt{1 + 4x^2}\, dx$$

(c) (5 pts.) Write a definite integral giving the exact *volume* of the solid generated by rotating the region about the x-axis. *Do not evaluate this integral*

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$$2\pi \int_0^1 [4 - (1 + x^2)^2]\, dx$$

(d) (7 pts.) Find the coordinates of the *center of mass* for this metal plate. Show your work.

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- $\overline{x} = 0$ (as the plate is symmetric about the y-axis)
- $\overline{y} = \dfrac{\text{Moment}}{\text{Mass}} \simeq \dfrac{10.667}{6.667} \simeq 1.6$ (units above the x-axis),

since

- ∘ Moment $= \displaystyle\int_1^2 5y(2\sqrt{y-1})\, dy = 10\int_1^2 y\sqrt{y-1}\, dy \simeq 10.667;$
- ∘ Mass $= \displaystyle\int_1^2 5(2\sqrt{y-1})\, dy = 10\int_1^2 \sqrt{y-1}\, dy \simeq 6.667.$

7. (3 points each) The following each require a short answer with no explanation.

(a) Give a function $f(x)$ so that the integral $\int_{-1}^{2} f(x)\,dx$ is an *improper* integral.

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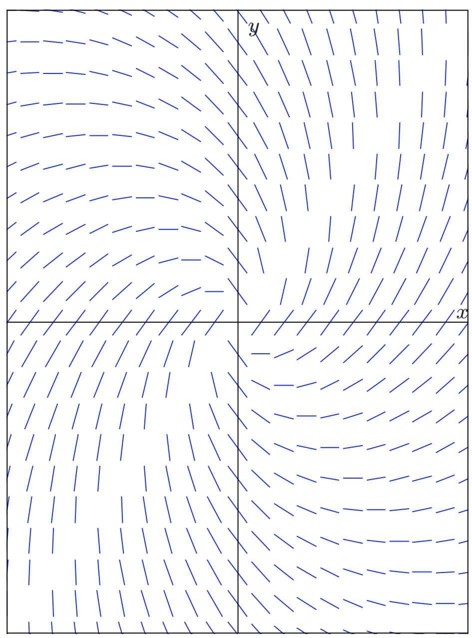
$$f(x) = \frac{1}{x}$$

(b) A slope field is shown below. Choose the differential equation that matches the given slope field.

(A) $\frac{dy}{dx} = x^2 - y^2$ (B) $\frac{dy}{dx} = y^2 - x^2$

$\boxed{\text{(C)}}$ $\frac{dy}{dx} = \frac{x+y}{x-y}$ (D) $\frac{dy}{dx} = \frac{x-y}{x+y}$



(This problem continues on the next page.)

(This is a continuation of Problem 7.)

(c) The graph of a function $f(x)$ is shown below. Which of the following could be the Taylor Polynomial approximating $f(x)$ for x near 0?

(A) $P_2(x) = 2 + 2x + 2x^2$

(B) $P_2(x) = 2 + 2x - 2x^2$

(C) $P_2(x) = 2 - 2x + 2x^2$

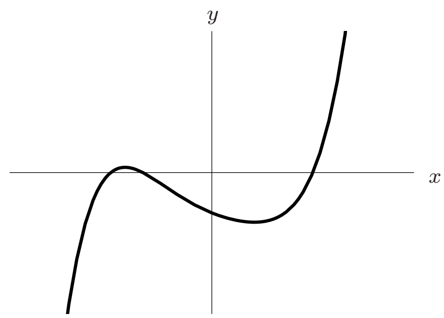
(D) $P_2(x) = 2 - 2x - 2x^2$

(E) $P_2(x) = -2 + 2x + 2x^2$

(F) $P_2(x) = -2 + 2x - 2x^2$

$\boxed{\text{(G)}}$ $P_2(x) = -2 - 2x + 2x^2$

(H) $P_2(x) = -2 - 2x - 2x^2$



(d) As a result of its operations, a nuclear power plant releases Cesium 137 at a rate of 0.1 millicuries (mCi) per year into the surrounding area. Cesium 137 is a short-lived radioactive isotope. It decays at a rate proportional to the amount of itself present, with constant of proportionality -0.0231. Assume that there is no other source of this particular isotope near the power plant. Write a differential equation satisfied by $Q(t)$, the quantity of Cesium 137 in mCi near this plant, where t is measured in years.

$$\frac{dQ}{dt} = 0.1 - 0.0231Q$$

8. (9 points) Recall that the Taylor series for $\cos x$ about $x = 0$ is given by $1 - x^2/2! + x^4/4! - x^6/6! + \cdots$.

   (a) (5 pts.) The Taylor series for $\cos x$ equals $\cos x$ wherever it converges. For which x-values does the Taylor series for $\cos x$ equal the function $\cos x$? Give a precise step-by-step argument that justifies your answer. *No graphs are allowed as justification.*

   For the given series we have:

   $$a_n = (-1)^n \frac{x^{2n-2}}{(2n-2)!}, \qquad a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)-2}}{(2(n+1)-2)!} = (-1)^{n+1} \frac{x^{2n}}{(2n)!},$$

   so then we get that

   - $\left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{x^{2n}}{(2n)!} \dfrac{(2n-2)!}{x^{2n-2}} = \dfrac{x^2}{(2n)(2n-1)},$ and
   - $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = x^2 \lim\limits_{n\to\infty} \dfrac{1}{(2n)(2n-1)} = 0.$

   So the radius of convergence is infinite, or convergence holds for *all* x-values (that is, all real numbers x.)

   (b) (4 pts.) Find all the solutions to the equation

   $$1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots = 0.$$

   You must show your work clearly and give exact answers. Calculator approximations or methods will receive no credit.

   We have,

   - $1 - \dfrac{(3x)^2}{2!} + \dfrac{(3x)^4}{4!} - \dfrac{(3x)^6}{6!} + \cdots = \cos(3x)$ and
   - $\cos(3x) = 0,$ which means: $3x = \pi/2 + k\pi,$

   or

   $$x = \frac{\pi}{6} + k\frac{\pi}{3}, \quad \text{where } k \text{ is any integer.}$$

9. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) A quantity x is distributed throughout a population with probability density function $p(x)$. If $p(10) = p(20)$, then none of the population has x values lying between 10 and 20.

TRUE $\boxed{\text{FALSE}}$

(b) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\boxed{\text{TRUE}}$ FALSE

(c) Let $P(x) = 1 - e^{-0.5x}$ for all $x \geq 0$ and $P(x) = 0$ otherwise. Then $P(x)$ could be a cumulative distribution function for some probability density function $p(x)$.

$\boxed{\text{TRUE}}$ FALSE

(d) $\displaystyle\int_3^x 2t\sin(t^2)dt$ is an antiderivative of $2x\sin(x^2)$.

$\boxed{\text{TRUE}}$ FALSE

(e) $y = \frac{1}{2}(\sin x - \cos x) + 2e^x$ is a solution to the differential equation $\frac{dy}{dx} = \cos x + y$.

$\boxed{\text{TRUE}}$ FALSE