Math 116 — Final Exam

December 18, 2006

Name: ____________________

Instructor: ____________________ Section: __________

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1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full keyboard). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	11	
2	7	
3	7	
4	10	
5	10	
6	8	
7	6	
8	10	
9	12	
10	7	
11	12	
Total	100	

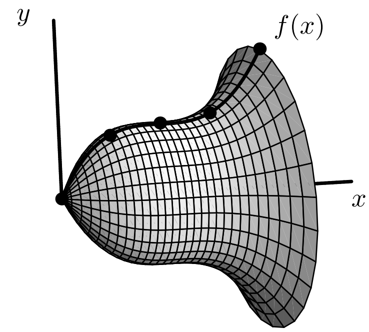
You may find it useful to note the following Taylor series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\,x^4 + \cdots$$



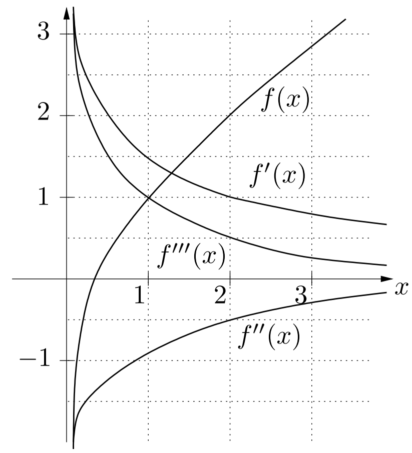
1. [11 points] Consider the shape shown to the right. The function shown as a dark curve is $f(x)$. The points on the curve are the points $(0, f(0))$, $(0.5, f(0.5))$, $(1, f(1))$, $(1.5, f(1.5))$, and $(2, f(2))$.

(a) [4 points of 11] Draw a slice, below, that you might use to find the total volume enclosed by the shape if you were to be doing this by integration. Label in your figure x, $f(x)$, Δx, and any other relevant quantities.

(b) [2 points of 11] Write an integral giving the volume of the shape.

(c) [5 points of 11] If the points shown in the figure are, in order from left to right, (0,0), (0.5,0.875), (1,1), (1.5,1.125) and (2,2), estimate the volume using the trapezoid method.

2. [7 points] The graph to the right shows $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$.



(a) [4 points of 7] Find the 3rd degree Taylor polynomial approximating $f(x)$ near $x = 2$.

(b) [3 points of 7] Based on the graphs of f and its derivatives that you have in the given figure, what would you guess the radius of convergence of the Taylor expansion for $f(x)$ around $x = 2$ would be? Explain.

3. [7 points] The function $\operatorname{erf}(x)$ is defined to be $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Find the Taylor series for $\operatorname{erf}(x)$ around $x = 0$.

4. [10 points] Consider the solution curve $y(x)$ to the differential equation $\frac{dy}{dx} = 1 + xy^2$ that goes through the point $(1, 1)$.

(a) [2 points of 10] Is this differential equation separable? Explain in one or two sentences.

(b) [5 points of 10] Use Euler's Method with three steps to approximate y when $x = 1.6$.

(c) [3 points of 10] Do you expect the real value of $y(1.6)$ to be greater than or less than the estimate you found in part (b)? Why?

5. [10 points] A mathematician proposed* that the velocity, $v(t)$, of a sprinter running less than 300 meters might satisfy the differential equation $\frac{dv}{dt} = k(v - R)$, for some constants k and R. For a sprint, it makes sense that $v(0) = 0$.

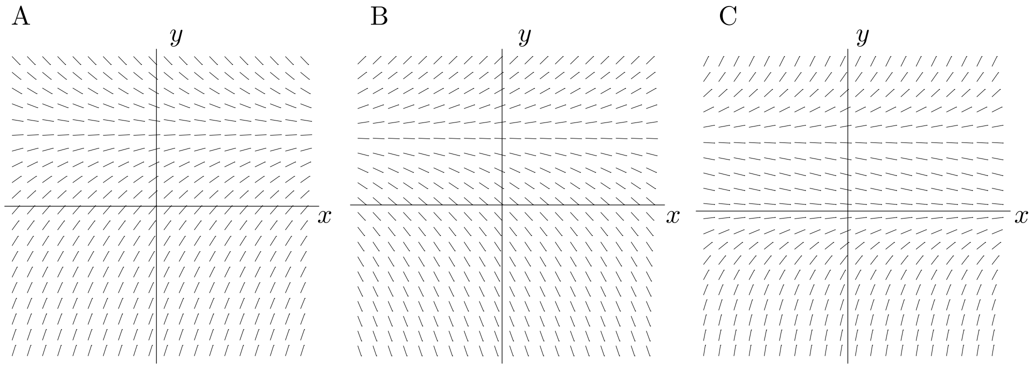
(a) [4 points of 10] Find the general solution to this differential equation.

(b) [2 points of 10] Find the particular solution to the initial value problem. (Your answer may involve the constants k and R.)

(c) [4 points of 10] Linford Christie won the men's 100 meter race in the 1993 World Track Championships in a time of 9.87 sec. If one second into the race he had reached 51% of his maximum possible speed, find values for the parameters k and R in the problem.

* *J.B. Keller, in Physics Today, Sept. 1973*

6. [8 points] Shown below are the slope fields for three differential equations, "A," "B," and "C." For each slope field, the axes intersect at the origin.



For each of the following functions, indicate which, if any, of the differential equations "A," "B," and "C" it could be the solution of. Note that any given function may be the solution to zero, one, or more than one of the differential equations. If a function is a solution to none of these differential equations, clearly write "None" as your answer.

(a) $y = 0$

(b) $y = 1$

(c) $y = 1 + ke^{x}$

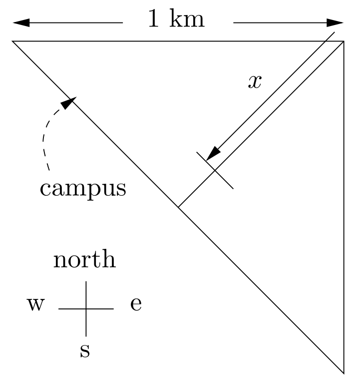
(d) $y = 1 + ke^{-x}$

7. [6 points] As part of a final project in a chemistry class, Alex is studying a reaction that combines a small amount of catalyst with a large amount of another reagent. The lab manual indicates that if the amount of reagent used is $R+x$, where R is the (large) intended amount and x is a small variation from that, then the amount of catalyst required is $c(x) = k\sqrt{R+x}$. However, Chris thinks that it would be reasonable (and easier!) to use $c(x) = k\sqrt{R}\,(1 + \frac{x}{2R})$ instead.

(a) [4 points of 6] Are Chris' and the lab book's expressions consistent? Explain. (Hint: your answer should not involve graphing.)

(b) [2 points of 6] Assuming that the two expressions are consistent, is Chris' estimate an over- or underestimate of the actual amount of catalyst required? Why?

8. [10 points] It turns out that there are a fair number of squirrels on the campus of Alex and Chris' university. The university occupies a triangular piece of land that is half of a 1 km by 1 km square, as shown in the figure to the right. Owing to the proximity of a local arboretum, the population of squirrels is densest at the northeast corner of campus. In addition, Alex has noted that if x is the distance measured along a line running diagonally northeast-southwest across the campus, as shown in the figure, the population density of squirrels everywhere along a line perpendicular to the diagonal is given by $p(x) = \frac{100}{x(1+x)}$ squirrels per square meter. How many squirrels are there on campus? (Note that there are 1000 meters in a kilometer.)



9. [12 points] It turns out that students at Alex and Chris' university have a strong tradition of taking university math classes. In fact, Chris determines that for the function $p(t) = \frac{1}{5(\frac{1}{5}+t)^2}$, the fraction of students having completed between t and $t + \Delta t$ years of collegiate mathematics is given approximately by $p(t)\,\Delta t$.

(a) [4 points of 12] Carefully find the fraction of students who have completed at least two years of university mathematics.

(b) [4 points of 12] Let $q(x)$ be the fraction of students that complete no more than x years of university mathematics. Write an integral that gives $q(x)$. Then evaluate your integral to find a formula for $q(x)$.

(c) [4 points of 12] We might think that the integral $\int_0^\infty t\,p(t)\,dt$ would give the average number of years of university mathematics that the students take. Explain why this does not make sense in this context. (Hint: how large is this value?)

10. [7 points] On the morning of December 1, Alex belatedly considered buying a plane ticket to go home the winter break. Unhappy with the price of the ticket, Alex decided to wait and see if the prices would drop at all, but, by checking with friends, determined that the rate of change in the price on December 1 was \$1.58/day. In the next two weeks, between days spent studying for calculus, Alex noted the rate of change in price, finding the data in the table below.

Date	Dec. 1	Dec. 4	Dec. 7	Dec. 10	Dec. 13	Dec. 16
Rate of Price Change (\$/day)	1.58	3.86	5.62	8.24	13.00	20.72

Admitting defeat, Alex bought a plane ticket on December 16. Give a good estimate for how much money Alex could have saved by buying the plane ticket on December 1.

11. [12 points] For each of the following, sketch a graph or figure as indicated. (There is more than one correct answer for each.) Your sketches do not have to be detailed, but should clearly illustrate the characteristics described.

(a) [3 points of 12] Sketch a non-constant function $f(x)$ such that neither of the left- or right-hand estimates for $\int_a^b f(x)\,dx$ are overestimates.

(b) [3 points of 12] Sketch the antiderivative of a function $f(x)$, on the interval $a \le x \le b$, if $\int_a^b f(x)\,dx < 0$.

(c) [3 points of 12] Sketch a function given in polar coordinates as $r = f(\theta)$, and the area represented by $\frac{1}{2}\left(f(\frac{\pi}{4})\right)^2 \Delta\theta$.

(d) [3 points of 12] Sketch a sequence S_n of partial sums for a convergent series $\Sigma\, a_n$ if $\Sigma\, a_n = L$.