Math 116 — First Exam

October 11, 2006

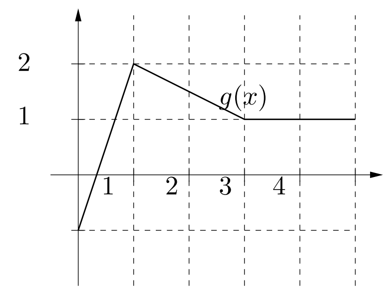
Name: Exam Solutions

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full keyboard). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	12	
4	8	
5	8	
6	10	
7	16	
8	12	
9	10	
Total	100	

1. [12 points] For all of parts (a)–(d), let $f(x) = 2x - 4$ and let $g(x)$ be given in the graph to the right.



(a) [3 points of 12] Find $\int_1^5 g'(x)\,dx$.

Solution:
By the Fundamental Theorem of Calculus,

$$\int_0^5 g'(x)\,dx = g(5) - g(1) = 1 - (2) = -1.$$

Alternately, note that $g'(x) = 3$ for $0 < x < 1$, $g'(x) = -\frac{1}{2}$ for $1 < x < 3$, and $g'(x) = 0$ for $x > 3$. Thus $\int_1^5 g'(x)\,dx = \int_1^3 -\frac{1}{2}\,dx = -1$.

(b) [3 points of 12] Find $\int_0^5 g(x)\,dx$.

Solution:
This integral is just the area between the graph of $g(x)$ and the x-axis between $x = 0$ and $x = 5$, counting area below the axis as negative. This is

$$\int_0^5 g(x)\,dx = -\text{ (area between 0 and 2/3)} + \text{(area between 0 and 1)} + \text{(area between 1 and 3)} +$$
$$\text{(area between 3 and 5)} = -\left(\frac{1}{2}(\frac{1}{3})(1)\right) + \left(\frac{1}{2}(\frac{2}{3})(2)\right) + (3) + (2) = 5\frac{1}{2}.$$

(c) [3 points of 12] Find $\int_2^{4.5} g(f(x))\,dx$.

Solution:
Using substitution with $w = f(x) = 2x - 4$, we have $\frac{1}{2}dw = dx$, so that $\int_2^{4.5} g(f(x))\,dx = \frac{1}{2}\int_{w(2)}^{w(4.5)} g(w)\,dw = \frac{1}{2}\int_0^5 g(w)\,dw$. But the calculation above gives $\int_0^5 g(w)\,dw = 5\frac{1}{2} = \frac{11}{2}$, so $\int_2^{4.5} g(f(x))\,dx = \frac{1}{2}\cdot\frac{11}{2} = \frac{11}{4} = 2.75$.

(d) [3 points of 12] Find $\int_0^5 f(x)\cdot g'(x)\,dx$.

Solution:
Using integration by parts with $u = 2x - 4$ and $v' = g'$, we have $u' = 2$ and $v = g$, so that

$$\int_0^5 f(x)\cdot g'(x)\,dx = (2x-4)\cdot g(x)\Big|_0^5 - 2\int_0^5 g(x)\,dx$$
$$= (6)(1) - (-4)(-1) - 2(\frac{11}{2}) = 2 - 11 = -9.$$

Alternate solution: note that $g'(x) = 3$ for $0 < x < 1$, $g'(x) = -\frac{1}{2}$ for $1 < x < 3$, and $g'(x) = 0$ for $x > 3$. Thus

$$\int_0^5 f(x)\cdot g'(x)\,dx = \int_0^1 3(2x-4)\,dx + \int_1^3 -\frac{1}{2}(2x-4)\,dx$$
$$= 3(x^2 - 4x)\Big|_0^1 + (-\frac{x^2}{2} + 2x)\Big|_1^3 = -9 + (\frac{3}{2} - \frac{3}{2}) = -9.$$

2. [12 points] While working on their team homework, Alex and Chris find that they have evaluated the same integral—but that they each used a different method, and got different answers! Alex found

$$\int (2x-1)(3+x)^4\,dx = (2x-1)\left(\frac{1}{5}(3+x)^5\right) - \frac{1}{15}(3+x)^6 + C.$$

while Chris had

$$\int (2x-1)(3+x)^4\,dx = \frac{1}{3}(3+x)^6 - \frac{7}{5}(3+x)^5 + C,$$

(a) [6 of 12 points] Considering the form of the solution that Alex found, what method is it most likely that Alex used? Use this method and verify that you obtain the same solution.

Solution:
We notice that the first term of Chris' solution, $(2x-1)(\frac{1}{5}(3+x)^5)$, is the product uv if $u = 2x-1$ and $v' = (3+x)^4$, so it looks as if this solution was obtained by using integration by parts. With these choices of u and v', we have $u' = 2$ and $v = \frac{1}{5}(3+x)^5$, so

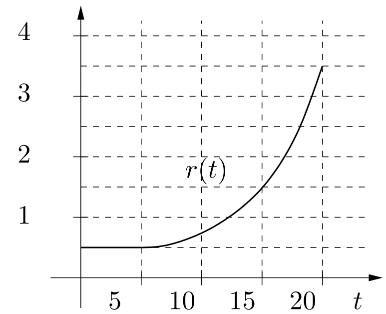
$$\begin{aligned}\int (2x-1)(3+x)^4\,dx &= (2x-1)\left(\frac{1}{5}(3+x)^5\right) - \int \frac{2}{5}(3+x)^5\,dx\\ &= (2x-1)\left(\frac{1}{5}(3+x)^5\right) - \frac{1}{15}(3+x)^6 + C.\end{aligned}$$

(b) [6 of 12 points] Considering the form of the solution that Chris found, what method is it most likely that Chris used? Use this method and verify that you obtain the same solution.

Solution:
We see only factors of $3+x$ to various powers in the solution, which suggests that Alex may have used substitution with $w = 3+x$. This works because then $dw = dx$ and $2x-1 = 2w-7$, so that

$$\begin{aligned}\int (2x-1)(3+x)^4\,dx &= \int (2w-7)w^4\,dw = \int 2w^5 - 7w^4\,dw\\ &= \frac{2}{6}w^6 - \frac{7}{5}w^5 + C = \frac{1}{3}(3+x)^6 - \frac{7}{5}(3+x)^5 + C.\end{aligned}$$

3. [12 points] Having completed their team homework, Alex and Chris are making chocolate chip cookies to celebrate. The rate at which they make their cookies, $r(t)$, is given in cookies/minute in the figure to the right (in which t is given in minutes). After $t =$ 20 minutes they have completed their cookie making extravaganza.



(a) [3 of 12 points] Write an expression for the total number of cookies that they make in the 20 minutes they are baking. Why does your expression give the total number of cookies?

Solution:
We are given the rate at which the cookies are being produced, so we know by the Fundamental Theorem of Calculus that the total number of cookies produced is given by $\int_0^{20} r(t)\,dt$.

(b) [3 of 12 points] Using $\Delta t = 5$, find left- and right-Riemann sum and trapezoid estimates for the total number of cookies that they make.

Solution:
Using $\Delta t = 5$, the left- and right-hand Riemann sums are

$$\begin{aligned} \text{LEFT}(4) &\approx 5(0.5 + 0.5 + 0.75 + 1.5) = 16.25 \\ \text{RIGHT}(4) &\approx 5(0.5 + 0.75 + 1.5 + 3.5) = 31.25. \end{aligned}$$

Thus the trapezoid estimate is $\text{TRAP}(4) = \frac{1}{2}(16.25 + 31.25) = 23.75$, or about 24 cookies.

(c) [3 of 12 points] How large could the error in each of your estimates in (b) be?

Solution:
We know that the maximum error in the left- or right-hand sums is just $\Delta t(r(20) - r(0)) = 5(3.5 - 0.5) = 15$ cookies. The maximum error in the trapezoid estimate is half this, or 7.5 cookies.

(d) [3 of 12 points] How would you have to change the way you found each of your estimates to reduce the possible errors noted in (c) to one quarter of their current values?

Solution:
The error in the left- or right-hand sums drops as n, so we would have to take four times as many steps, reducing Δt to $\frac{5}{4} = 1.25$ min to reduce the error in those estimates by a factor of four. The error in the trapezoid estimate drops as n^2, the square of the number of steps we take in the calculation, so if we recalculated our estimate with $\Delta t = 2.5$ minutes it would be four times as accurate.

4. [8 points] Use the fact that $\int_0^\infty e^{-x}\sin(x)\,dx = \frac{1}{2}$ to find $\int_0^\infty e^{-x}\cos(x)\,dx$.

Solution:
We use integration by parts on $\int_0^\infty e^{-x}\cos(x)\,dx$ with $u = e^{-x}$, so that $u' = -e^{-x}$ and $v' = \cos(x)$, so that $v = \sin(x)$. Then

$$\begin{aligned}\int_0^\infty e^{-x}\cos(x)\,dx &= \lim_{b\to\infty}\left(e^{-x}\sin(x)\Big|_0^b + \int_0^b e^{-x}\sin(x)\,dx\right)\\ &= \lim_{b\to\infty}\left(e^{-b}\sin(b)\right) + \int_0^\infty e^{-x}\sin(x)\,dx\\ &= 0 + \frac{1}{2}.\end{aligned}$$

Thus $\int_0^\infty e^{-x}\cos(x)\,dx = \frac{1}{2}$.
Alternate solution: with $u = \sin(x)$ and $v' = e^{-x}$, we have $u' = \cos(x)$ and $v = -e^{-x}$, so that $\int_0^\infty e^{-x}\sin(x)\,dx = \lim_{b\to\infty}(-\cos(b)\,e^{-b} + 1) - \int_0^\infty e^{-x}\sin(x)\,dx = 1 - \frac{1}{2} = \frac{1}{2}$.
Second alternate solution: Use integration by parts twice to solve for answer without using the given $\int_0^\infty e^{-x}\sin(x)\,dx = \frac{1}{2}$. Note that this doesn't completely follow the instructions, and therefore cannot receive full credit.

5. [8 points] Let $F(x) = \int_0^{x^2(x-1)} g(t)\,dt$, where $g(t)$ is always positive. For what values of x is $F(x)$ increasing? For what values is it decreasing?

Solution:
$F(x)$ is increasing when $F'(x) > 0$ and decreasing when $F'(x) < 0$. Taking the derivative, $F'(x) = \frac{d}{dx}\int_0^{x^2(x-1)} g(t)\,dt = (2x(x-1) + x^2)\cdot g(x^2(x-1))$. Because g is always positive, the sign of this expression is determined by the first term, $2x(x-1) + x^2 = 3x^2 - 2x$. We can see when this is positive and negative by graphing it, or by factoring. Factoring, $3x^2 - 2x = x(3x-2)$, which is negative for $0 < x < \frac{2}{3}$, and positive for $x < 0$ and $x > \frac{2}{3}$. Thus $F(x)$ is increasing for $x < 0$ and $x > \frac{2}{3}$, and decreasing for $0 < x < \frac{2}{3}$.

6. [10 points] While eating cookies, Alex notes that the velocity of a student passing by is given, in meters/second, by the data shown below.

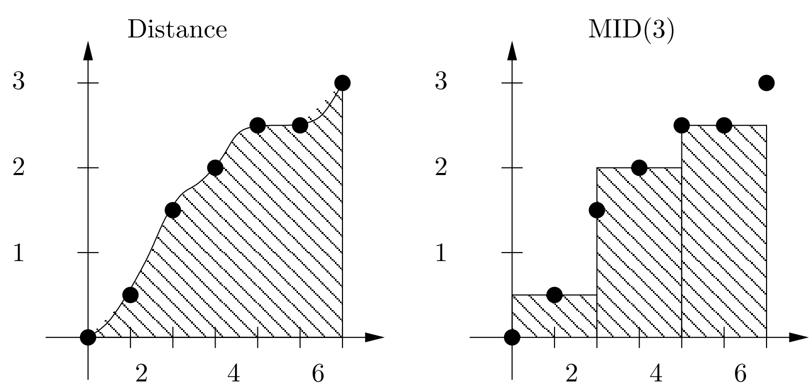
t (seconds)	0	1	2	3	4	5	6
$v(t)$ (m/s)	0	0.5	1.5	2	2.5	2.5	3

(a) [5 of 10 points] Using the midpoint rule, find as accurate an estimate as possible for the total distance the student travels in the six seconds shown in the table (use only the given data in your calculation).

Solution:
The smallest value of Δt we can use with the given data is $\Delta t = 2$ seconds; if we try to use $\Delta t = 1$, we need values of v at $t = \frac{1}{2}$, $\frac{3}{2}$, etc., which aren't available in the given data. Then, using $\Delta t = 2$, we have

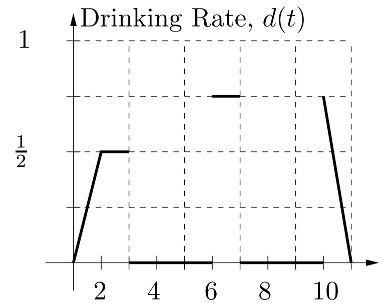
$$\text{MID}(3) = 2(0.5 + 2 + 2.5) = 10 \text{ meters.}$$

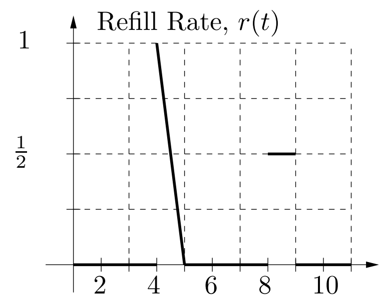
(b) [5 of 10 points] Draw two figures on the axes below, one each to illustrate the total distance you are estimating and the estimate you found. Be sure it is clear how your figures illustrate the indicated quantities.



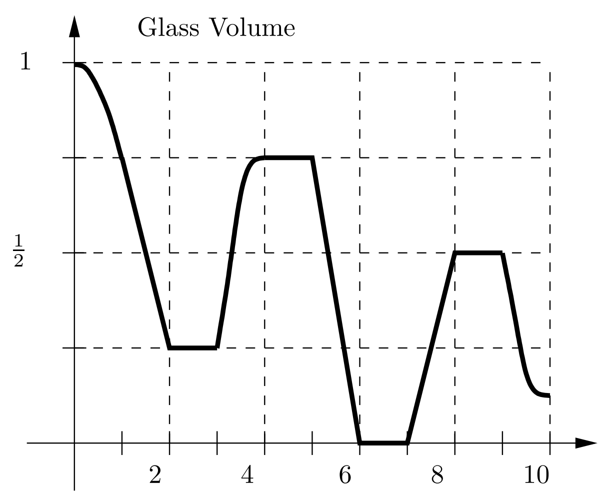
Solution:
We show the actual distance traveled and MID(3) in the figures above right. The actual distance traveled is the shaded area under the function $v(t)$ between $t = 0$ and $t = 6$; MID(3) is the shaded area in the indicated rectangles.

7. [16 points] As they eat their cookies, Alex and Chris are drinking milk. They each have a glass from which they are drinking, and the jug of milk is conveniently located in the middle of the table. They both start with a full glass of milk (one pint). As they are drinking, Chris sneakily refills the glass that Alex is drinking from while Alex is distracted by the student that is passing by. The rate at which Alex is drinking the milk, $d(t)$, is shown in the top figure to the right, while the rate at which Chris refills the glass, $r(t)$, is shown in the bottom figure. Both figures give the rates in pints per minute, and the time in minutes.





(a) [8 of 16 points] Let $V(t)$ be the amount (volume) of milk in the glass that Alex is drinking from, as a function of time. Carefully sketch $V(t)$ on the axes provided. Be sure to label your axes and that your sketch has accurate vertical and horizontal scales.



Solution:
To find the amount of milk in the glass, start with the initial volume ($V = 1$ pint), and use the Fundamental Theorem of Calculus to calculate the change in volume given the rates. The area under the top rate curve (the rate at which Alex is drinking) results in a decrease in the volume of the glass, while the area under the bottom (the rate at which Chris is refilling the glass) increases the volume. This results in the bottom graph shown above

(b) [2 of 16 points] Does Alex ever finish all of the milk in the glass?

Solution:
Alex empties the glass at time $t = 6$ minutes, but Chris refills it to half full starting at $t = 7$.

7. *continued: see the previous page for any information you need about the problem.*

(c) [4 of 16 points] Give, but do not evaluate, an expression that gives the average amount of milk in Alex' glass for the time interval $0 \le t \le 10$. Your expression may use any of the functions $d(t)$, $r(t)$ and $V(t)$, etc., given in the problem.

Solution:
The average volume in the glass is given by

$$\text{average volume} = \frac{1}{10}\int_0^{10} V(t)\, dt,$$

where $V(t)$ is the volume that we found and graphed in (a).

(d) [2 of 16 points] If you had to evaluate your expression in (c), how would you do it? (Explain in one or two sentences.)

Solution:
To find the average value we need to find $\int_0^{10} \text{volume}\, dt$. This is just the area under the volume graph, so we need to find or estimate the area under the volume graph. The easiest way to calculate this would be to draw a grid on our graph and estimate the area by counting the grid boxes that lie below the graph. We could also find an estimate by using the trapezoid method, which would probably be the most accurate of the estimates we could find.

Alternate solution: Note that the volume is the antiderivative of $f(t) - d(t)$, where $f(t)$ is the rate at which the glass is filled (the lower graph) and $d(t)$ is the rate at which the glass is emptied (the upper graph). Both of these are piecewise linear, so we could find (piecewise) expressions for them. Then $V(t)$, the volume in the glass, is just the antiderivative of $f(t) - d(t)$ with $V(0) = 1$, which we could also find explicitly. And the average value is one-tenth of the integral of this $V(t)$ from $t = 0$ to $t = 10$. But this seems like a rather painful way of approaching the solution.

8. [12 points] In class, Chris' calculus professor is well known to cover material at a rate $m(t) = \frac{1}{12(t-20)^{2/3}}$ textbook sections/minute, where t is the time in minutes since the start of class.

(a) [2 of 12 points] What is the meaning of the integral $\int_0^{80} m(t)\,dt$ (include units in your explanation)?

Solution:
The integral is the area under the rate $m(t)$ between $t = 0$ and $t = 80$, which, by the Fundamental Theorem of Calculus is the total number of sections that Chris' professor covers in the 80 minute (90 less ten) class period.

(b) [4 of 12 points] How many sections would you estimate the professor covers in the first minute of class? In the 20th minute? Why?

Solution:
We note that $m(0) = \frac{1}{12(20)^{2/3}} = 0.0113$ sections/minute. Thus we might guess that the professor might cover approximately 0.0113 sections-worth of material in the first minute. If we repeat this calculation for the 20th minute, we might guess that the number of sections covered is $(1\,\text{minute})(m(19)) = \frac{1}{12(1)^{2/3}} = \frac{1}{12}$ sections. Note, however, that $m(20)$ is undefined—therefore, it appears that the professor is speaking at an infinite rate at about $t = 20$, so that we might also wonder if an infinite number of words are spoken. We can verify this by completing part (c) of this problem.

(c) [6 of 12 points] Find exactly (that is, by hand) the value of $\int_0^{80} m(t)\,dt$.

Solution:
We note that because $m(t)$ is discontinuous at $t = 20$ this is an improper integral. We therefore evaluate

$$\begin{aligned}
\int_0^{80} \frac{1}{12(t-20)^{2/3}}\,dt &= \lim_{a\to 20^-} \int_0^{a} \frac{1}{12(t-20)^{2/3}}\,dt + \lim_{a\to 20^+} \int_a^{80} \frac{1}{12(t-20)^{2/3}}\,dt \\
&= \lim_{a\to 20^-} \frac{1}{4}(t-20)^{1/3}\Big|_0^a + \lim_{a\to 20^+} \frac{1}{4}(t-20)^{1/3}\Big|_a^{80} \\
&= \lim_{a\to 20^-} \frac{1}{4}\left((a-20)^{1/3} + (20)^{1/3}\right) + \lim_{a\to 20^+} \frac{1}{4}\left((60)^{1/3} - (a-20)^{1/3}\right) \\
&= \frac{1}{4}(20)^{1/3} + \frac{1}{4}(60)^{1/3}\ \text{sections.}
\end{aligned}$$

Or, approximately 1.66 sections per class period. Because this integral is finite, it is clear that the amount of material covered in the 20th minute is, in fact, finite too.

9. [10 points] To improve their understanding of the material in their Calculus course, Alex and Chris have invented a set of statements about the material they have been studying. These statements are given below. For each statement, circle **true** (that is, the statement is always true), or **false** (it isn't), and give a one sentence explanation for your answer.

(a) [2 of 10 points] If a bounded continuous function $f(x)$ has the properties that $f(x) > \frac{1}{x}$ for $1 < x < 50$, $f(50) = \frac{1}{50}$, and $f(x) < \frac{1}{x}$ for $x > 50$, then $\int_1^\infty f(x)\,dx$ converges.

TRUE FALSE

Solution:
False. The behavior of $f(x)$ for $x \leq 50$ doesn't matter for the convergence of $\int_1^\infty f(x)\,dx$, but because $\int_1^\infty \frac{1}{x}\,dx$ diverges, $f(x) < \frac{1}{x}$ for $x > 50$ only tells us that the area under $f(x)$ is guaranteed to be "less infinite" than that under $\frac{1}{x}$.

(b) [2 of 10 points] If a bounded continuous function $f(x)$ has the properties that $f(x) > \frac{1}{x^2}$ for $1 < x < 50$, $f(50) = \frac{1}{2500}$, and $f(x) < \frac{1}{x^2}$ for $x > 50$, then $\int_1^\infty f(x)\,dx$ converges.

TRUE FALSE

Solution:
True. Using the same logic as in (a), we know that the area under $f(x)$ is less than the area under $\frac{1}{x^2}$; thus, because $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so too must $\int_1^\infty f(x)\,dx$. As a side note, this argument assumes that $f(x) \geq 0$, which should have been included in the conditions on $f(x)$ given in the problem.

(c) [2 of 10 points] Since the function $\frac{\sin(x)+2}{\sqrt{x}}$ is always less than $\frac{3}{\sqrt{x}}$ for $2 \leq x < \infty$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, we know that $\int_2^\infty \frac{\sin(x)+2}{\sqrt{x}}\,dx$ converges.

TRUE FALSE

Solution:
False. Again, while $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, $\int_2^\infty \frac{2}{\sqrt{x}}\,dx$ diverges, so we know only that $\int_2^\infty \frac{\sin(x)+1}{\sqrt{x}}\,dx$ is smaller than that infinite value, which provides no guarantee that it converges.

(d) [2 of 10 points] If $0 < \frac{1}{x} < g(x) < \frac{1}{x^2}$ for $0 < x < 1$, then the area between $g(x)$ and the x-axis for $0 < x < 1$ is guaranteed to be finite.

TRUE FALSE

Solution:
False. We know that both $\int_0^1 \frac{1}{x}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$ diverge, so the integral of $g(x)$ must also diverge, and the area is correspondingly infinite.

(e) [2 of 10 points] Let $f(x) = \frac{1}{(x-1)^2}$. Then if $F(x) = \int_0^x f(t)\,dt$, we know that $F(0) = 0$ and that $F(2) = \int_0^2 \frac{1}{(t-1)^2}\,dt = -\frac{1}{t-1}\Big|_0^2 = -1 - 1 = -2$.

TRUE FALSE

Solution:
False. Because $\frac{1}{(x-1)^2}$ is undefined at $x = 1$, we need to evaluate $F(2)$ as an improper integral, which because the integrand looks like $(x-1)^{-2}$ near $x = 1$ will diverge.