Math 116 — Second Exam

November 15, 2006

Name: Exam Solutions

Instructor: ______________________ Section: ________

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1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full keyboard). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	10	
3	14	
4	12	
5	8	
6	16	
7	12	
8	16	
Total	100	

1. [12 points] While at home for Thanksgiving, Alex finds a forgotten can of corn that has been sitting on the shelf for a number of years. The contents have started to settle towards the bottom of the can, and the density of corn inside the can is therefore a function, $\delta(h)$, of the height h (measured in cm) from the bottom of the can. δ is measured in g/cm^3. The can has a radius of 4 cm, and a height of 12 cm.

(a) [3 points of 12] Write an expression that approximates the mass of corn in the cylindrical cross-section from height h to height $h + \Delta h$.

Solution:
The cylindrical cross-section is a disk with height Δh and radius 4 cm, so its volume is $\Delta V = \pi(4)^2 \cdot \Delta h$. The mass of the cross-section is then $\Delta M = \delta(h) \cdot \Delta V = 16\pi \cdot \delta(h) \cdot \Delta h$.

(b) [3 points of 12] Write a definite integral that gives the total mass of corn in the can.

Solution:
We let $\Delta h \to 0$ and add the contributions from each disk by integrating, to get

$$M = \int_0^{12} 16\pi \cdot \delta(h)\, dh.$$

(c) [3 points of 12] If $\delta(h) = 4e^{-0.03h}$, what is the total mass of corn inside the can?

Solution:
We have $M = \int_0^{12} 16\pi \cdot \delta(h)\, dh = \int_0^{12} 16\pi \cdot 4e^{-0.03h}\, dh$. Thus

$$M = 64\pi \int_0^{12} e^{-0.03h}\, dh = -\frac{64}{0.03}\pi\, e^{-0.03h}\Big|_0^{12} = -\frac{6400}{3}\pi\left(e^{-0.36} - 1\right) \approx 2026.2 \text{ g}.$$

(d) [3 points of 12] Write, but do not evaluate, an expression for the can's center of mass in the h direction. Would you expect the center of mass to be in the top or bottom half of the can? Do not solve for the center of mass, but in one sentence, justify your answer.

Solution:
The center of mass is

$$\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{M},$$

where M is the mass we found before. We expect this to be in the bottom half of the can, because the density decreases with increasing h. (Obviously, we could also write $\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{\int_0^{12} 16\pi \cdot \delta(h)}$; plugging in $\delta(h)$ from (c) is fine too.)

2. [10 points] Alex and Chris decide to invest money in a savings account to prepare for their expenses after they land a posh mathematical consulting job following their success in calculus. They deposit \$100 on the first of each month into an account that pays 0.4167% interest at the end of each month (an annual yield of about 5%). Let B_n be the amount in their account immediately after their nth deposit.

(a) [5 points of 10] B_n is a sequence. Give the first four terms in this sequence.

Solution:
After the first deposit, Alex and Chris have

$$B_1 = \$100.$$

Immediately before the second deposit, they get 0.4167% interest on this, and so have \$100.42, to which they add \$100. Thus

$$B_2 = \$200.42(= (1.004167)(100) + 100).$$

Similarly,

$$\begin{aligned} B_3 &= \$200.42(1.004167) + \$100 = \$301.26, \quad \text{and} \\ B_4 &= \$301.26(1.004167) + \$100 = \$402.52. \end{aligned}$$

(b) [5 points of 10] Write a general, closed-form, formula for B_n (your expression should involve none of the symbols Σ, $\cdots$, or $\int$).

Solution:
If we rewrite the preceding slightly, we can see that B_n is just the sum of a geometric series. We have

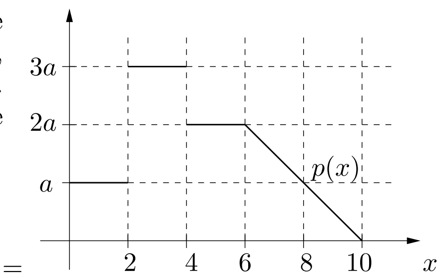
$$\begin{aligned} B_1 &= \$100, \\ B_2 &= \$100(1.004167) + \$100, \\ B_3 &= (\$100(1.004167) + \$100)(1.004167) + \$100 \\ &= \$100(1.004167)^2 + \$100(1.004167) + \$100, \end{aligned}$$

etc. Thus $B_n = \$100(1.004167^{n-1}) + \cdots + \$100(1.004167) + \$100$. This is a finite geometric series with n terms, and so

$$B_n = \$100\left(\frac{1-(1.004167)^n}{1-1.004167}\right) \approx \$23,998.08\,(1.004167^n - 1)\,.$$

(Either of these is fine as the correct answer.)

3. [14 points] For the graduating class of 2010 from a major university (its name concealed so as to protect its identity), the probability density function, $p(x)$, for the number of job offers, x, obtained by a graduate is shown in the figure to the right. The value a appearing in the values on the y-axis of this figure is a constant.



(a) [3 points of 14] What is the value of a?

Solution:
We know that $\int_0^\infty p(x)\,dx = 1$, so we must have $\int_0^{10} p(x)\,dx = 1$. Calculating the value of the integral using the area under the curve, $\int_0^{10} p(x)\,dx = 16a$, so that $16a = 1$, or $a = \frac{1}{16}$.

(b) [3 points of 14] What is the probability that a graduate will get at least 4 but no more than 8 job offers?

Solution:
This is just $\int_4^8 p(x)\,dx$. We can find the actual value for this probability by evaluting the area under the curve, which is $\frac{7}{16}$.

(c) [4 points of 14] Write, but do not evaluate, an expression giving the mean number of job offers obtained by a graduate. Explain in one sentence how you would evaluate your expression.

Solution:
The mean number of offers is $\overline{x} = \int_0^{10} x \cdot p(x)\,dx$. To find this we would have to find a piecewise expression for $p(x)$ (for $0 < x < 2$, $p(x) = \frac{1}{16}$, etc.), multiply each by x, and evaluate the resulting integral(s).

(d) [4 points of 14] Write an expression that gives the median number of job offers obtained by a graduate. Use your expression to find the median.

Solution:
The median number of offers, T, is the number such that half of the graduates get less than or equal to T offers. This is T so that $0.5 = \int_0^T p(x)\,dx$, which requires us to find the value T such that the area under $p(x)$ from $x = 0$ to $x = T$ is the same as the area under $p(x)$ from $x = T$ to $x = 10$. By inspection of the areas shown in the figure, this is $x = 4$.

4. [12 points] The following three parts of this problem have to do with the convergence of series.

(a) [4 points of 12] For $\Sigma \frac{5}{1+n+e^n}$:

i. [2 points of 4] What is a good test to determine the convergence of this series? Explain, in 1–2 sentences only, why this is.

Solution:
The Comparison Test is a good test for this series. This is because we know the convergence of $\Sigma \frac{5}{e^n}$, and note that $\frac{5}{1+n+e^n} < \frac{5}{e^n}$, so we have a ready made comparison to test the convergence of $\Sigma \frac{5}{1+n+e^n}$. Limit comparison will also work (which is reasonable because there's an obvious comparison series), as will the ratio test (which is reasonable because of the exponential term).

ii. [2 points of 4] Determine if this series converges, diverges, or if we can't tell.

Solution:
Because $\frac{5}{1+n+e^n} < \frac{5}{e^n}$, and because we know that $\Sigma \frac{5}{1+n+e^n}$ converges, we know that $\Sigma \frac{5}{1+n+e^n}$ must also converge. Alternately, limit comparison with the same two series gives $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^n}{1+n+e^n} = 1$, a finite non-zero number, so both must converge. And also alternately, the ratio test gives $\lim_{n\to\infty} |\frac{1+n+e^n}{1+n+e^{n+1}}| = \frac{1}{e} < 1$, so, again, the series must converge. Math is astonishingly consistent.

(b) [4 points of 12] For $\Sigma \frac{n}{n^2+5}$:

i. [2 points of 4] What is a good test to determine the convergence of this series? Explain, in 1–2 sentences only, why this is.

Solution:
The Integral Test is a good test for this series. This is because it is easy to integrate $\frac{x}{x^2+5}$, so the integral test is a dandy choice. The limit comparison test is another good option, because $\frac{n}{n^2+5} \sim \frac{n}{n^2} = \frac{1}{n}$ when $n \to \infty$, which makes us think this series must diverge, but because $\frac{n}{n^2+5} < \frac{n}{n^2}$ the comparison test doesn't work when comparing to $\frac{1}{n}$. It's possible to use the comparison test, but that requires quite a bit of cunning to use correctly.

ii. [2 points of 4] Determine if this series converges, diverges, or if we can't tell.

Solution:
Integrating, $\int_1^\infty \frac{x}{x^2+5}\,dx = \lim_{b\to\infty} \frac{1}{2}\left(\ln(b^2+5) - \ln(6)\right)$, which diverges as $b \to \infty$, so by the integral test the series $\Sigma \frac{n}{n^2+5}$ must also diverge. Alternately, using the limit comparison test with the two series $\Sigma \frac{n}{n^2+5}$ and $\Sigma \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{n}{n^2+5} \cdot \frac{n}{1} = 1$, so that, knowing that $\Sigma \frac{1}{n}$ diverges, we must have that our series diverges.

this problem is continued on the following page. . .

. . . problem continued from the previous page.

(c) [4 points of 12] For $\Sigma \frac{n}{2n^3-1}$:

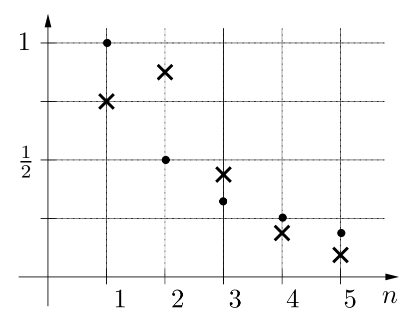
i. [2 points of 4] What is a good test to determine the convergence of this series? Explain, in 1–2 sentences only, why this is.

Solution:
The Limit Comparison Test is a good test for this series. This is because as $n \to \infty$, we note that $\frac{n}{2n^3-1} \sim \frac{n}{2n^3} = \frac{1}{2n^2}$, which makes us think this series must converge. However, $\frac{n}{2n^3-1} > \frac{1}{2n^2}$, so the comparison test doesn't work when comparing to $\frac{1}{2n^2}$, which leads us to try the limit comparison test. It's possible to use the comparison test, but requires quite a bit of cunning to use correctly.

ii. [2 points of 4] Determine if this series converges, diverges, or if we can't tell.

Solution:
We choose to compare with $\Sigma \frac{1}{n^2}$. The ratio of the terms $\frac{n}{2n^3-1}$ and $\frac{1}{n^2}$ is $\frac{n^3}{2n^3-1}$, so that as $n \to \infty$ the ratio is 1. Thus, knowing that $\Sigma \frac{1}{n^2}$ converges, we conclude by the limit comparison test that $\Sigma \frac{n}{2n^3-1}$ must also converge.

5. [8 points] Let a_n and b_n be the two sequences shown in the figure to the right. The sequence $a_n = \frac{1}{n}$ is shown with solid dots (•) and the sequence b_n is shown with crosses (×). For $5 \le n < \infty$, $0 < b_n < a_n$.



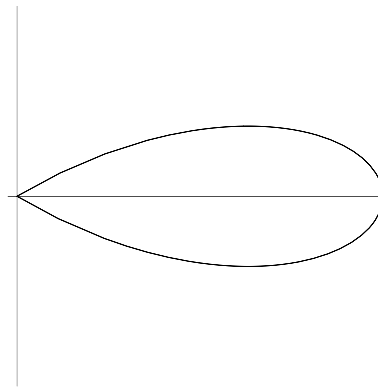
(a) [4 points of 8] Does the sequence b_n converge, diverge, or can we not tell? Explain in one or two sentences. If it converges, indicate the value to which it converges.

Solution:
We know that the sequence $a_n = \frac{1}{n}$ converges to zero, and for large n the b_n are trapped between a_n and zero, so b_n must also converge to zero.

(b) [4 points of 8] Does the series $\Sigma\, b_n$ converge, diverge, or can we not tell? Explain in one or two sentences. If it converges, indicate the value to which it converges.

Solution:
We know that $0 < b_n < a_n$ for n large, but all this tells us is that partial sums of $\Sigma\, a_n$, which is a divergent series, are larger than those of $\Sigma\, b_n$. Thus we are unable to determine whether $\Sigma\, b_n$ converges or diverges.

6. [16 points] Chris has decided to take flying lessons, and notices that the cross-section of the airplane wing is given approximately by the figure to the right. The front-to-back length of the wing, as shown in the figure, is 2 m. The end-to-end length of the wing is 15 m (that is, its length along an axis coming out of this page is 15 m), and its ends are flat.



(a) [3 points of 16] If this cross-section is described by the polar equation $r = a\cos(3\theta)$, what is a?

Solution:
The indicated front-to-back length occurs when $\theta = 0$. $r(0) = a$, so we must have $a = 2$ m.

(b) [4 points of 16] What range of values for θ generate this figure?

Solution:
The curve starts and ends at $r = 0$, which requires that $\cos(3\theta) = 0$, so that $3\theta = \pm\frac{\pi}{2}$ is a good solution. Thus $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. There are many other sets of θ values that give the region as well (e.g., any interval that gives the top half of the wing, e.g., $[0, \frac{\pi}{6}]$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{4\pi}{3}, \frac{3\pi}{2}]$, $[2\pi, \frac{13\pi}{6}]$, etc., plus any interval that gives the bottom half, e.g., $[-\frac{\pi}{6}, 0]$, $[\frac{\pi}{2}, \frac{2\pi}{3}]$, $[\frac{7\pi}{6}, \frac{4\pi}{3}]$, $[\frac{11\pi}{6}, 2\pi]$, etc.).

(c) [9 points of 16] Airplanes frequently have fuel tanks in their wings. If 75% of the wing's volume is available space for a fuel tank, what volume of fuel could be stored in this wing?

Solution:
The volume of the wing is given by the area of its cross-section times 15 m. Slicing with polar slices, a slice of the cross-sectional area is given by $\Delta A \approx \frac{1}{2}r^2\,\Delta\theta = \frac{1}{2}(2\cos(3\theta))^2\,\Delta\theta$, so that the total cross-sectional area is given by $\int_{-\pi/6}^{\pi/6} 2\cos^2(3\theta)\,d\theta = \int_{-\pi/6}^{\pi/6} 1+\cos(6\theta)\,d\theta = (\theta+\frac{1}{6}\sin(6\theta))\big|_{-\pi/6}^{\pi/6} = \frac{\pi}{3}$. Thus the total volume is 5π m^3, and the total volume available for fuel storage is $\frac{15\pi}{4}$ m^3, or about 11.8 m^3.

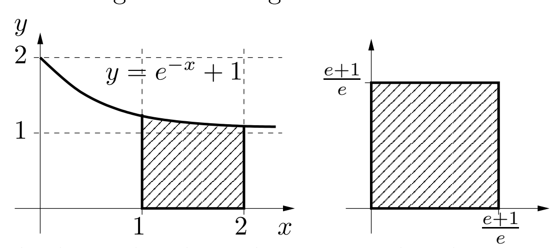
7. [12 points] A mysterious three-dimensional abstract sculpture has appeared on the major university's central campus. Alex, being a particularly astute calculus student, notes that the volume is given by $V = \int_1^2 (e^{-x} + 1)^2\, dx$, where x is in meters.

(a) [4 points of 12] What does the integrand of Alex' integral tell you about the shape of the sculpture?

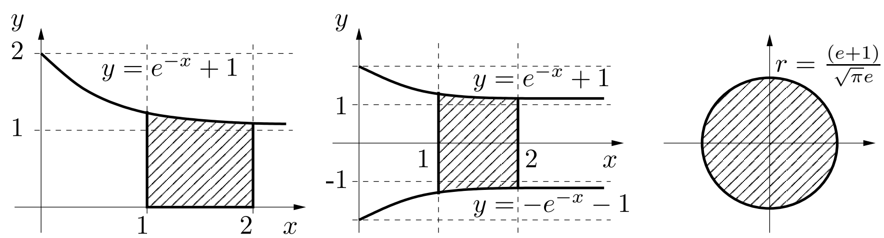
Solution:
It appears that Alex considered a "slice" of the sculpture that has volume $\Delta V = (e^{-x} + 1)^2\, \Delta x$. Thus $(e^{-x} + 1)^2$ appears to be the cross-sectional area of the slice—suggesting that the cross-sections of the sculpture perpendicular to the x-axis are square. There are several other correct interpretations of this, the most obvious of which is that the object has rotational symmetry about the x-axis and has circular cross-sections with radius $r = \frac{1}{\sqrt{\pi}}(e^{-x} + 1)$.

(b) [4 points of 12] Suppose that the sculpture was placed on a set of x-y axes. Sketch the base of the sculpture, labeling all important dimensions and features.

Solution:
If we follow the first "slicing" indicated in (a), with square cross-sections, it is reasonable to guess that the base is bounded by $1 \le x \le 2$ and $0 \le y \le e^{-x}+1$, as shown in the figure to the left, below. Alternately, if we regard x as a dummy variable that measures up the object, we may consider the base to be as shown in the figure to the right.

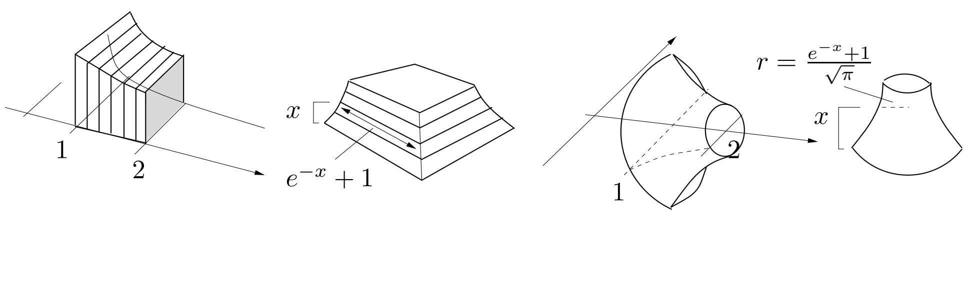


Similarly, if we think about the object having circular slices, we can think of the "base" of the object in any of the following three ways: the first two look at the projection of the object into an xy-plane, thinking of the x in the integral as that measured along the indicated x-axis, and the last considers x to be a dummy variable that measures up the object.

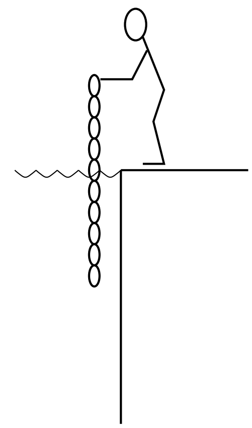


(c) [4 points of 12] Sketch and/or carefully explain what the shape of the sculpture is.

Solution:
Sketched below are, from left to right, the figure with square cross-sections on the base $1 \le x \le 2$, $0 \le y \le e^{-x} + 1$; the figure with square cross-sections and x measuring up the figure; The figure with circular cross-sections rotated around the x-axis, and the figure with circular cross sections and x measuring height.



8. [16 points] Chris is standing at the edge of a swimming pool, holding a chain that is partially submerged in the water of the pool, as shown in the figure to the right. The chain is six feet long and weighs 5 lb/ft. When it is in the water, however, the buoyant force of the water makes the effective weight of the chain less—in the water, it weighs only 3 lb/ft. If the chain is initially half submerged in the pool and Chris lifts it straight up until it is entirely out of the water, how much work does Chris do?



Solution:
We can find the total work by considering the work to move from a given position, x (measured as the distance that the chain has been raised), to the position $x + \Delta x$. The required force is the weight of the chain,

$$F = (\text{weight of chain above the water}) + (\text{weight of chain in the water}).$$

The length of chain above the water is $3 + x$ ft, and the length below is $3 - x$ ft. Thus $F = (3 + x)(5) + (3 - x)(3) = 24 + 2x$ lb. The work to lift the chain through this distance Δx is then $\Delta W = (24+2x)\Delta x$. The total work is found by integrating over the 3 ft that it is lifted, so

$$W = \int_0^3 (24 + 2x)\,dx = (24x + x^2)\Big|_0^3 = 72 + 9 = 81\,\text{ft}\cdot\text{lb}.$$

An alternate solution is to consider the top half and bottom half of the chain separately. The top half all moves 3 ft and has a constant weight, so the work is $W_t = ((3\,\text{ft})(5\,\text{lb/ft}))(3\,\text{ft}) = 45\,\text{ft}\cdot\text{lb}$. Then, a piece of the bottom part of chain that has length Δx and is x feet from the bottom of the chain's initial position has a weight 3 lb/ft for the distance $3 - x$ and a weight 5 lb/ft for the distance x. Thus the work to lift it is $\Delta W = (3\Delta x)(3-x) + (5\Delta x)x = (9+2x)\Delta x$. The total work to lift the bottom half of the chain is then $W_b = \int_0^3 9 + 2x\,dx = 9x + x^2\big|_0^3 = 27 + 9 = 36\,\text{ft}\cdot\text{lb}$. The total work is the sum of W_t and W_b, which is not surprisingly still 81 ft·lb.