Math 116 — Final Exam

December 18, 2006

Name: Exam Solutions

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full keyboard). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	11	
2	7	
3	7	
4	10	
5	10	
6	8	
7	6	
8	10	
9	12	
10	7	
11	12	
Total	100	

You may find it useful to note the following Taylor series:

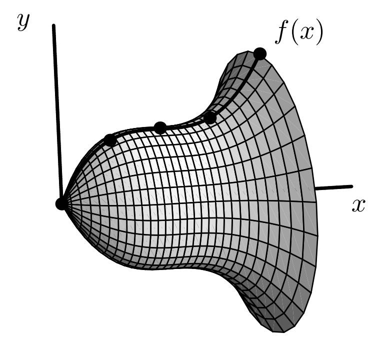
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

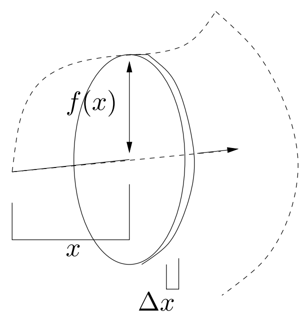
$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\,x^4 + \cdots$$

1. [11 points] Consider the shape shown to the right. The function shown as a dark curve is $f(x)$. The points on the curve are the points $(0, f(0))$, $(0.5, f(0.5))$, $(1, f(1))$, $(1.5, f(1.5))$, and $(2, f(2))$.



(a) [4 points of 11] Draw a slice, below, that you might use to find the total volume enclosed by the shape if you were to be doing this by integration. Label in your figure x, $f(x)$, Δx, and any other relevant quantities.

Solution:
We are given the function $f(x)$, which provides the radius of circular cross-sections of the object shown. Thus we slice vertically, getting slices that are disks, as shown in the figure below.



(b) [2 points of 11] Write an integral giving the volume of the shape.

Solution:
The volume of the slice is $\Delta V = \pi\,(f(x))^2\,\Delta x$, so that, letting Δx go to zero, we can sum all such slices with an integral. The resulting volume is $V = \int_0^2 \pi\,(f(x))^2\,dx$.

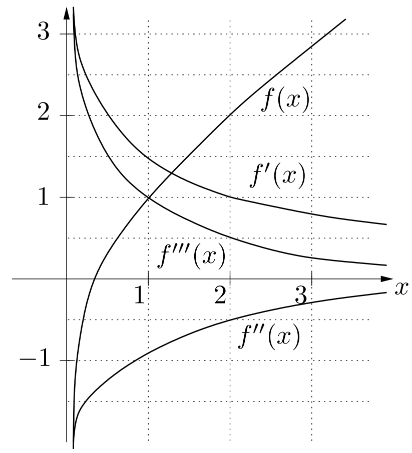
(c) [5 points of 11] If the points shown in the figure are, in order from left to right, (0,0), (0.5,0.875), (1,1), (1.5,1.125) and (2,2), estimate the volume using the trapezoid method.

Solution:
Left- and right-hand sums for the volume are

$$\text{Left} = (0.5)(\pi(0)^2 + \pi(0.875)^2 + \pi(1)^2 + \pi(1.125)^2) \approx 4.761$$
$$\text{Right} = (0.5)(\pi(0.875)^2 + \pi(1)^2 + \pi(1.125)^2 + \pi(2)^2) \approx 11.04$$

Then the trapezoid method gives $\text{Trap} = \frac{1}{2}(4.761 + 11.04) = 7.90$.

2. [7 points] The graph to the right shows $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$.



(a) [4 points of 7] Find the 3rd degree Taylor polynomial approximating $f(x)$ near $x = 2$.

Solution:
We know that the 3rd degree Taylor polynomial is $P_3 = f(2) + f'(2)(x-2) + \frac{1}{2!}f''(2)(x-2)^2 + \frac{1}{3!}f'''(2)(x-2)^3$. We can read the values for f and its derivatives from the graphs, finding $f(x) = 2$, $f'(2) = 1$, $f''(2) = -\frac{1}{2}$ and $f'''(2) = \frac{1}{2}$. Thus

$$P_3 = 2 + (x-2) - \frac{1}{4}(x-2)^2 + \frac{1}{12}(x-2)^3.$$

(b) [3 points of 7] Based on the graphs of f and its derivatives that you have in the given figure, what would you guess the radius of convergence of the Taylor expansion for $f(x)$ around $x = 2$ would be? Explain.

Solution:
From the graphs it is clear that f and its derivatives have a vertical asymptote at $x = 0$. It is not possible for a polynomial expansion to reproduce this, so we would expect that the Taylor expansion would fail there. This is two units from $x = 2$, so we guess that the radius of convergence is $R = 2$.

3. [7 points] The function $\mathrm{erf}(x)$ is defined to be $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$. Find the Taylor series for $\mathrm{erf}(x)$ around $x = 0$.

Solution:
We know that the Taylor series for e^t at $t = 0$ is $e^t = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!}$. Thus the series for $e^{-t^2} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n\, t^{2n}}{n!}$.
We can integrate this to find the Taylor series for $\mathrm{erf}(x)$:

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n}}{n!}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n\, t^{2n}}{n!}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)\,n!}.$$

Note that if we try and derive this from first principles it is more difficult to get the general term: $\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$, so $\mathrm{erf}''(x) = \frac{2}{\sqrt{\pi}}(-2xe^{-x^2})$, and so on: $\mathrm{erf}'''(x) = \frac{2}{\sqrt{\pi}}(-2e^{-x^2} + 4x^2e^{-x^2})$, $\mathrm{erf}^4(x) = \frac{2}{\sqrt{\pi}}(12xe^{-x^2} - 8x^3e^{-x^2})$, etc. So $\mathrm{erf}(0) = 0$, $\mathrm{erf}'(0) = \frac{2}{\sqrt{\pi}}$, $\mathrm{erf}''(0) = 0$, $\mathrm{erf}'(0) = \frac{2}{\sqrt{\pi}} \cdot (-2)$, $\mathrm{erf}^4(0) = 0$, etc. This gives us the first two non-zero terms of the series, but doesn't shed much insight on the general progression.

4. [10 points] Consider the solution curve $y(x)$ to the differential equation $\frac{dy}{dx} = 1 + xy^2$ that goes through the point $(1, 1)$.

(a) [2 points of 10] Is this differential equation separable? Explain in one or two sentences.

Solution:
This is not separable. In order to separate variables we have to be able to get all of the "y"s on one side of the equation, multiplying the dy, and all of the "x"s on the other, multiplying the dx. Because of the additive factor of one on the right-hand side this is not possible.

(b) [5 points of 10] Use Euler's Method with three steps to approximate y when $x = 1.6$.

Solution:
If we're taking three steps, our step size is $\Delta x = 0.2$. At $(x, y) = (1, 1)$ the slope is $\frac{dy}{dx} = 2$, so we estimate $y(1.2) \approx 1 + 0.2(2) = 1.4$. Continuing, we obtain the following table of values:

x	y	dy/dx	so that
1	1	2	$y(1.2) \approx 1 + 0.2(2) = 1.4$
1.2	1.4	$1 + (1.2)(1.4)^2 = 3.352$	$y(1.4) \approx 1.4 + 0.2(3.352) = 2.0704$
1.4	2.0704	$1 + (1.4)(2.0704)^2 \approx 7.001$	$y(1.6) \approx 2.0704 + 0.2(7.001) \approx 3.471$

Thus, we estimate that $y(1.6) \approx 3.471$.

(c) [3 points of 10] Do you expect the real value of $y(1.6)$ to be greater than or less than the estimate you found in part (b)? Why?

Solution:
We note that the slopes that we generated in the approximations (column 3 in the table above) are increasing. We therefore expect that the Euler estimates will undershoot the actual values at every step, and that the estimate $y(1.6) \approx 3.471$ is an underestimate.

5. [10 points] A mathematician proposed* that the velocity, $v(t)$, of a sprinter running less than 300 meters might satisfy the differential equation $\frac{dv}{dt} = k(v - R)$, for some constants k and R. For a sprint, it makes sense that $v(0) = 0$.

(a) [4 points of 10] Find the general solution to this differential equation.

Solution:
We can separate variables to get $\frac{dv}{v-R} = k dt$. Integrating both sides, we have $\ln|v - R| = kt + C$, so that $v = R + Ae^{kt}$, for some arbitrary constant $A = \pm e^C$.

(b) [2 points of 10] Find the particular solution to the initial value problem. (Your answer may involve the constants k and R.)

Solution:
We know that $v(0) = 0$. This gives $0 = R + A$, so $A = -R$, and the particular solution is $v = R(1 - e^{kt})$.

(c) [4 points of 10] Linford Christie won the men's 100 meter race in the 1993 World Track Championships in a time of 9.87 sec. If one second into the race he had reached 51% of his maximum possible speed, find values for the parameters k and R in the problem.

Solution:
Note that $v = R$ is an equilibrium solution to the equation, and that as $t \to \infty$ we must have $v \to R$ (k is negative, so that $\frac{dv}{dt} > 0$ when $v = 0$). Thus Christie's top speed is R, and if Christie reaches 51% of his maximum speed after one second, we know $v(1) = 0.51R$. Thus $R(1 - e^k) = 0.51R$, so that $e^k = 0.49$, and $k = \ln(0.49) \approx -0.713$. As we expected, this is less than zero. Then we know that in 9.87 seconds Christie covers 100 m. This is the integral of the velocity, so that

$$\begin{aligned} 100 = \int_0^{9.87} R(1 - e^{-0.713t})\, dt &= R\left(t + \frac{1}{0.713}\, e^{-0.713t}\right)\Bigg|_0^{9.87} \\ &= R\left(9.87 + 1.403\,(e^{-7.037} - 1)\right) \\ &= 8.468R. \end{aligned}$$

Dividing by 8.468, we find $R = \frac{100}{8.468} = 11.81$.

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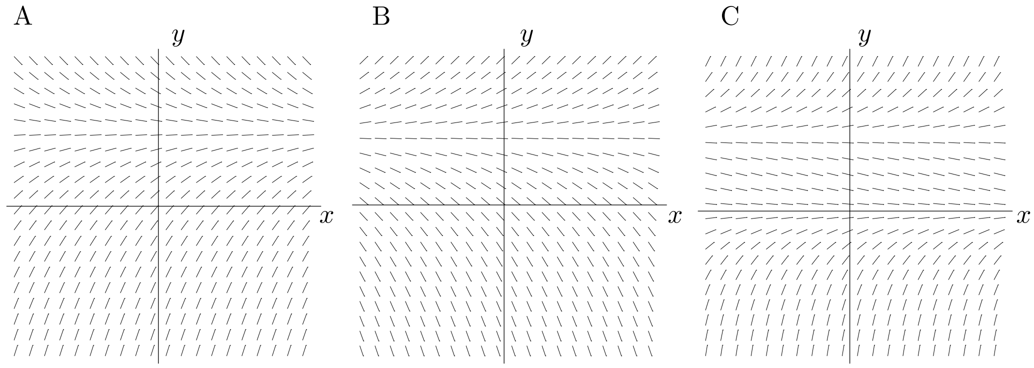
* *J.B. Keller, in Physics Today, Sept. 1973*

6. [8 points] Shown below are the slope fields for three differential equations, "A," "B," and "C." For each slope field, the axes intersect at the origin.



For each of the following functions, indicate which, if any, of the differential equations "A," "B," and "C" it could be the solution of. Note that any given function may be the solution to zero, one, or more than one of the differential equations. If a function is a solution to none of these differential equations, clearly write "None" as your answer.

(a) $y = 0$

Solution:
Answer: "C." This is a constant, so any differential equation(s) it solves must have the equilibrium solution $y = 0$. This means that the slope is zero for any x when $y = 0$, which is true only for differential equation "C." Thus $y = 0$ is a solution only to "C."

(b) $y = 1$

Solution:
Answer: "A, B, C." Again, this is a constant, so any differential equation(s) it solves must have the equilibrium solution $y = 1$. This means that the slope is zero for any x when $y = 0$, which is true for all three differential equations—if the horizontal slopes in the figures are at $y = 1$. Thus $y = 1$ could be a solution to any of "A," "B," or "C."

(c) $y = 1 + ke^{x}$

Solution:
Answer: "B." We note that if $k = 0$ we get the constant solution $y = 1$. If $k < 0$, we must be starting with the initial condition $y(0) < 1$. For all such cases, the exponential grows and as $x \to \infty$, $y \to -\infty$. We can see that for differential equation "B" this is the case: following the slope field lines from any point on the y axis below $y = 1$ gives a solution that decreases faster and faster. For "A" and "C" this is not the case—while "C" has decreasing solutions for $0 < y(0) < 1$, for $y(0) < 0$ they increase. Thus this solution could solve "B" only.

(d) $y = 1 + ke^{-x}$

Solution:
Answer: "A." As in the previous problem, if $k = 0$ we get the constant solution $y = 1$. If $k < 0$, we must be starting with the initial condition $y(0) < 1$. For all such cases, the exponential decays to zero and as $x \to \infty$, $y \to 1$. We can see that for differential equation "A" this is the case: following the slope field lines from any point on the y axis below $y = 1$ gives an increasing solution that approaches one. For "B" and "C" this is not the case. Thus this could solve "A" only.

7. [6 points] As part of a final project in a chemistry class, Alex is studying a reaction that combines a small amount of catalyst with a large amount of another reagent. The lab manual indicates that if the amount of reagent used is $R+x$, where R is the (large) intended amount and x is a small variation from that, then the amount of catalyst required is $c(x) = k\sqrt{R+x}$. However, Chris thinks that it would be reasonable (and easier!) to use $c(x) = k\sqrt{R}\,(1+\frac{x}{2R})$ instead.

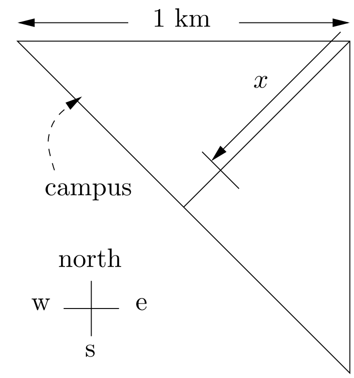
(a) [4 points of 6] Are Chris' and the lab book's expressions consistent? Explain. (Hint: your answer should not involve graphing.)

Solution:
They are consistent (of course). If we start with the lab book's expression and factor $\sqrt{R}$ out of the square root, we get $c(x) = k\sqrt{R}\sqrt{1+\frac{x}{R}}$. Then $\frac{x}{R}$ is small, so we can logically expand this as a Taylor series for small $\frac{x}{R}$ with the binomial expansion. This gives $c(x) = k\sqrt{R}\,(1+\frac{x}{2R}-\frac{x^2}{8R^2}+\cdots)$. Thus the Chris' expression is the same as the lab book's up to the $\frac{x}{R}$ term.

(b) [2 points of 6] Assuming that the two expressions are consistent, is Chris' estimate an over- or underestimate of the actual amount of catalyst required? Why?

Solution:
The next term in the binomial expansion is $-\frac{x^2}{8R^2}$, which decreases the value of $c(x)$ (and, because R is large, subsequent terms are smaller), so Chris' estimate is an overestimate.

8. [10 points] It turns out that there are a fair number of squirrels on the campus of Alex and Chris' university. The university occupies a triangular piece of land that is half of a 1 km by 1 km square, as shown in the figure to the right. Owing to the proximity of a local arboretum, the population of squirrels is densest at the northeast corner of campus. In addition, Alex has noted that if x is the distance measured along a line running diagonally northeast-southwest across the campus, as shown in the figure, the population density of squirrels everywhere along a line perpendicular to the diagonal is given by $p(x) = \frac{100}{x(1+x)}$ squirrels per square meter. How many squirrels are there on campus? (Note that there are 1000 meters in a kilometer.)



Solution:
Because the squirrel population is constant along the perpendicular lines, we want to slice the campus into slices along these perpendiculars. Each slice has width Δx, and is a distance x along the indicated northwest–southeast line. Note that $0 \le x \le \frac{1000}{\sqrt{2}}$. Then the length of the slice is $2x$, and its area is $2x\,\Delta x$. The number of squirrels on the slice is then $\rho(x) \cdot 2x\,\Delta x = \frac{200}{1+x}\,\Delta x$. We can find the total number of squirrels by integrating this over all possible values of x, finding

$$\#\text{ squirrels} = \int_0^{1000/\sqrt{2}} \frac{200}{(1+x)}\,dx = 200\,\ln(1+x)\Big|_0^{1000/\sqrt{2}} = 200\,\ln(1+\frac{1000}{\sqrt{2}}) \approx 1312.52.$$

A fractional squirrel seems a bit morbid, so let's assume that there are 1313 squirrels.

9. [12 points] It turns out that students at Alex and Chris' university have a strong tradition of taking university math classes. In fact, Chris determines that for the function $p(t) = \frac{1}{5(\frac{1}{5}+t)^2}$, the fraction of students having completed between t and $t + \Delta t$ years of collegiate mathematics is given approximately by $p(t)\,\Delta t$.

(a) [4 points of 12] Carefully find the fraction of students who have completed at least two years of university mathematics.

Solution:
Given the property that $p(t)\,\Delta t$ gives the fraction of students having completed between t and $t + \Delta t$ years of collegiate mathematics, we can find the fraction having completed at least two years of mathematics by integrating. This is $\int_2^\infty \frac{1}{5(\frac{1}{5}+t)^2}\,dt$. This is clearly an improper integral, so we evaluate it with some care and a limit. $\int_2^\infty \frac{1}{5(\frac{1}{5}+t)^2}\,dt = \lim_{b\to\infty} \int_2^b \frac{1}{5(\frac{1}{5}+t)^2}\,dt = \lim_{b\to\infty} (-\frac{1}{5(\frac{1}{5}+b)} + \frac{1}{5(\frac{1}{5}+2)}) = \frac{1}{11}$. Or, about 9%.

(b) [4 points of 12] Let $q(x)$ be the fraction of students that complete no more than x years of university mathematics. Write an integral that gives $q(x)$. Then evaluate your integral to find a formula for $q(x)$.

Solution:
We note that $q(x) = \int_0^x p(t)\,dt$, an antiderivative of $p(t)$. Evaluating, we get $q(x) = 1 - \frac{1}{5(\frac{1}{5}+x)} = 1 - \frac{1}{1+5x} = \frac{5x}{1+5x}$.

(c) [4 points of 12] We might think that the integral $\int_0^\infty t\,p(t)\,dt$ would give the average number of years of university mathematics that the students take. Explain why this does not make sense in this context. (Hint: how large is this value?)

Solution:
Note that for $t \geq 1$, $\frac{t}{5(\frac{1}{5}+t)^2} > \frac{t}{5(t+t)^2} = \frac{1}{20t}$, and $\int_1^\infty \frac{1}{20t}\,dt$ diverges. Thus $\int_1^\infty \frac{t}{5(\frac{1}{5}+t)^2}\,dt$ diverges, which means that $\int_0^\infty \frac{t}{5(\frac{1}{5}+t)^2}\,dt$ must also. This suggests that the mean number of years of university mathematics that the students study is infinite, which seems unlikely.

10. [7 points] On the morning of December 1, Alex belatedly considered buying a plane ticket to go home the winter break. Unhappy with the price of the ticket, Alex decided to wait and see if the prices would drop at all, but, by checking with friends, determined that the rate of change in the price on December 1 was \$1.58/day. In the next two weeks, between days spent studying for calculus, Alex noted the rate of change in price, finding the data in the table below.

Date	Dec. 1	Dec. 4	Dec. 7	Dec. 10	Dec. 13	Dec. 16
Rate of Price Change (\$/day)	1.58	3.86	5.62	8.24	13.00	20.72

Admitting defeat, Alex bought a plane ticket on December 16. Give a good estimate for how much money Alex could have saved by buying the plane ticket on December 1.

Solution:
The total change in the price of the plane ticket is $\int_1^{16}(\text{rate of change})\,dt$. We can estimate this integral using a left- or right-hand sum. These give

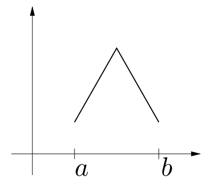
$$\text{Left sum} = (3)(1.58 + 3.86 + 5.62 + 8.24 + 13) = \$96.90$$
$$\text{Right sum} = (3)(3.86 + 5.62 + 8.24 + 13 + 20.72) = \$154.32.$$

Because it appears that the rate is an increasing function, we expect that the left sum is an underestimate and the right an overestimate, so that a better guess would be the trapezoid estimate $\text{Trap} = \frac{1}{2}(96.90 + 154.32) = \$125.61$.

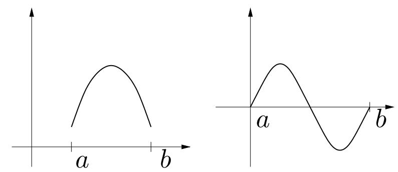
11. [12 points] For each of the following, sketch a graph or figure as indicated. (There is more than one correct answer for each.) Your sketches do not have to be detailed, but should clearly illustrate the characteristics described.

(a) [3 points of 12] Sketch a non-constant function $f(x)$ such that neither of the left- or right-hand estimates for $\int_a^b f(x)\,dx$ are overestimates.

Solution:

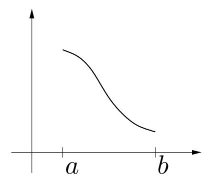


Or, e.g.,



(b) [3 points of 12] Sketch the antiderivative of a function $f(x)$, on the interval $a \le x \le b$, if $\int_a^b f(x)\,dx < 0$.

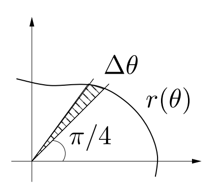
Solution:



Key: $F(b)$ must be less than $F(a)$.

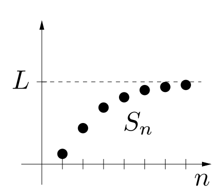
(c) [3 points of 12] Sketch a function given in polar coordinates as $r = f(\theta)$, and the area represented by $\frac{1}{2}\left(f(\frac{\pi}{4})\right)^2 \Delta\theta$.

Solution:



(d) [3 points of 12] Sketch a sequence S_n of partial sums for a convergent series $\Sigma\, a_n$ if $\Sigma\, a_n = L$.

Solution:



Other figures are also ok, e.g.,