Math 116 — First Exam

October 10, 2007

Name: ____________________

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a partial table of integrals, and useful integrals for comparison of improper integrals, on the first page of the exam (after this cover page). Any integrals or comparisons other than these should be explicitly derived.

Problem	Points	Score
1	16	
2	14	
3	10	
4	10	
5	12	
6	12	
7	14	
8	12	
Total	100	

You may find the following partial table of integrals to be useful:

$$\int e^{ax}\,\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\sin(bx) - b\cos(bx) + C),$$

$$\int e^{ax}\,\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos(bx) + b\sin(bx) + C),$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + C, a \neq b,$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + C, a \neq b,$$

$$\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + C, a \neq b.$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$

$$\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln(x^2+a^2) + \frac{c}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C, a \neq b$$

$$\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}((ac+d)\ln|x-a| - (bc+d)\ln|x-b|) + C, a \neq b$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

You may use the convergence properties of the following integrals:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$

$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges for } p < 1 \text{ and diverges for } p \geq 1,$$

$$\int_0^\infty e^{-ax}\,dx \text{ converges for } a > 0.$$

1. [16 points] For this problem, $\int_1^5 g(x)\,dx = 12$, and $f(x) = 2x - 9$. Values of $g(x)$ are given in the table below.

x	1	2	3	4	5
$g(x)$	0.1	1.5	2	5	10

(a) [5 points of 16] Find $\int_5^7 g(f(x))\,dx$

(b) [5 points of 16] Find $\int_1^5 f(x) \cdot g'(x)\,dx$.

(c) [6 points of 16] Find $\int_1^5 \frac{g'(x)}{g(x)(g(x)+1)}\,dx$.

2. [14 points] Consider the integral $\int_0^2 \frac{(\sin\sqrt{x})+1}{\sqrt{x}}\,dx$.

(a) [2 points of 14] Explain why this is an improper integral.

(b) [6 points of 14] Carefully show, using an appropriate comparison function (that is, without actually evaluating the integral), that this integral converges.

(c) [6 points of 14] Carefully work out the actual value of the integral.

3. [10 points] The Great Pyramid of Giza in Egypt was originally (approximately) 480 ft high. Its base was originally (again, approximately) a square with side lengths 760 ft.

(a) [6 points of 10] Sketch a slice that could be used to calculate the volume of the pyramid by integrating. In your sketch, indicate all variables you are using. Find an expression for the volume of the slice in terms of those variables.

(b) [4 points of 10] Use your slice from (a) to find the volume of the pyramid.

4. [10 points] The Belgian scholar Lambert Quetelet published a distribution of chest measurements of Scottish soldiers in 1846. His distribution showed that the expected probability of a soldier having a chest measurement between 38 and 40 inches was given approximately by

$$P = \frac{1}{\sqrt{2\pi}} \int_{-0.9}^{0.1} e^{-y^2/2}\, dy.$$

Suppose that a scholar studying 19th century Scottish soldiers' physical measurements estimates this probability using the following calculation:

$$P = \frac{1}{\sqrt{2\pi}} \int_{-0.9}^{0.1} e^{-y^2/2}\, dy \approx \frac{1}{\sqrt{2\pi}} (0.25)(0.7406 + 0.8713 + 0.9629 + 0.9997) = 0.3565.$$

(a) [6 points of 10] What numerical method did the scholar use, how many steps were used in the method, and what was the step size?

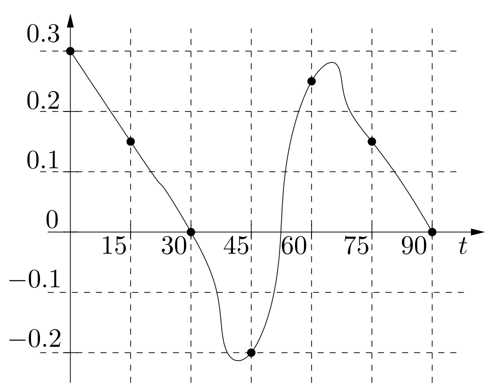
(b) [4 points of 10] Sketch a graph of $e^{-y^2/2}$. Explain how your graph indicates whether the scholar's approximation is an over- or under-estimate for the actual value of the integral.

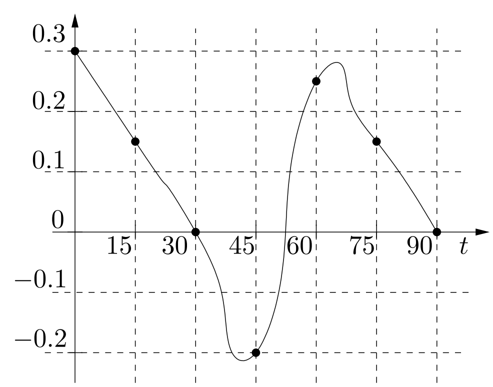
5. [12 points] Suppose that a student's rate of exam completion, given in problems per minute, is given by

t	0	15	30	45	60	75	90
$r(t)$	0.30	0.15	0	−0.20	0.25	0.15	0

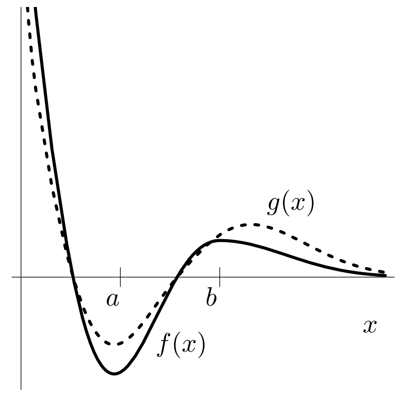
(a) [6 points of 12] Write an integral that gives the total number of problems that the student completes in the 90 minute exam period. Estimate the number of problems that the student completes. Based on your estimate, does the student complete the exam?

(b) [6 points of 12] The points on $r(t)$ given in (a) are shown in the two graphs below, connected by a smooth curve. On the first graph, illustrate the geometric meaning of (i) $\int_{30}^{45} r(t)\,dt$ and (ii) $\frac{1}{30}\int_0^{30} r(t)\,dt$. On the second graph, illustrate the geometric meaning of your calculation in (a).





6. [12 points] Consider the graphs of $f(x)$ (the solid curve) and $g(x)$ (the dashed curve) in the figure to the right. Suppose that $\int_0^a f(x)\,dx$ converges, and that $\int_b^\infty f(x)\,dx$ diverges. Assume that the behavior suggested as $x \to 0$ and $x \to \infty$ continues for that part of the range $0 \le x < \infty$ not shown in the graph.



For each of the following indicate what this graph suggests about the convergence of the indicated integral. Circle one answer only for each part. No explanation is necessary.

(a) [2 points of 12] $\int_0^b g(x)\,dx$

converges diverges its convergence cannot be determined

(b) [2 points of 12] $\int_a^\infty g(x)\,dx$

converges diverges its convergence cannot be determined

(c) [2 points of 12] $\int_0^\infty g(x)\,dx$

converges diverges its convergence cannot be determined

(d) [2 points of 12] $\int_a^b g(x)\,dx$

converges diverges its convergence cannot be determined

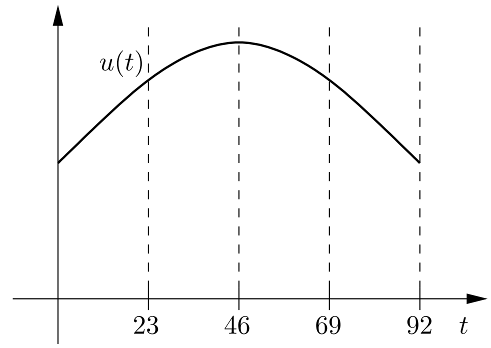
(e) [2 points of 12] $\int_0^\infty (f(x) + g(x))\,dx$

converges diverges its convergence cannot be determined

(f) [2 points of 12] $\int_0^\infty (f(x) - g(x))\,dx$

converges diverges its convergence cannot be determined

7. [14 points] Suppose that the rate of electricity use, $u(t)$, by a large calculus-problem producing factory is given, in megawatt-hours per day for the three months of summer, by the graph to the right. Time t is given in days from the beginning of the summer. As shown, the function $u(t)$ is symmetric about the middle of summer, July 16, which falls 46 days from the beginning of the summer. Let $M(T)$ be the *average rate of electricity use* of the factory over the course of the first T days of the summer.



(a) [6 points of 14] Write a formula for $M(T)$ in terms of $u(t)$.

(b) [8 points of 14] Fill in the missing blanks of the following table of values of $M(T)$. Show your work, so that it is clear how you obtained your answers.

T	23	46	69	92
$M(T)$	8			11

8. [12 points] The velocity of an object, with air resistance, may in some circumstances be given as

$$v(t) = \sqrt{\frac{g}{k}} \left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1} \right),$$

where g is the acceleration due to gravity, k is a constant representing air resistance, and $m = \sqrt{gk}$.

(a) [2 points of 12] Write an expression for the distance D that the object falls in the first t_0 seconds.

(b) [5 points of 12] Find the distance D *(note that half of this calculation is significantly harder than the rest; do not waste too much time on it if you get stuck).*

(c) [5 points of 12] Suppose $\sqrt{g/k} = 10$ and $m = 1$. Note that in this case $v(3) = 9.95 \approx 10$. Use a geometric argument to show that the distance traveled between $t = 0$ and $t = 3$, $D(3)$, satisfies the inequality $15 < D(3) < 30$.