Math 116 — Second Exam

November 14, 2007

Name: ____

Instructor: ______________________________ Section: ________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. There is a partial table of integrals, and useful integrals for comparison of improper integrals and series, on the first page of the exam (after this cover page).

Problem	Points	Score
1	6	
2	6	
3	16	
4	12	
5	16	
6	12	
7	9	
8	8	
9	15	
Total	100	

You may find the following partial table of integrals to be useful:

$$\int e^{ax}\,\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\sin(bx) - b\cos(bx) + C),$$
$$\int e^{ax}\,\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos(bx) + b\sin(bx) + C),$$
$$\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + C, a \neq b,$$
$$\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + C, a \neq b.$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln(x^2+a^2) + \frac{c}{a}\arctan(\frac{x}{a}) + C, a \neq 0$$
$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C, a \neq b$$
$$\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}((ac+d)\ln|x-a| - (bc+d)\ln|x-b|) + C, a \neq b$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C$$
$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

You may use the convergence properties of the following integrals and series:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$
$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges for } p < 1 \text{ and diverges for } p \geq 1,$$
$$\int_0^\infty e^{-ax}\,dx \text{ converges for } a > 0,$$
$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p > 1 \text{ and diverges for } p \leq 1,$$
$$\sum_{n=1}^\infty \frac{1}{a^n} \text{ converges for } a > 1 \text{ and diverges for } a \leq 1.$$

1. [6 points] Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n^5)}$$

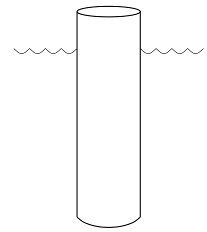
converges or diverges. Explain your reasoning.

2. [6 points] Use one of the ratio or limit comparison tests to determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3^n - 2}}$$

converges. Explain your reasoning.

3. [16 points] A cylindrical buoy with a radius of 0.2 m and height 2 m floats with 20% of its height above water, as suggested by the figure to the right. If the density of water is $\delta = 1000$ kg/m^3 and the acceleration due to gravity is 9.81 m/s^2, find the force on the exterior of the buoy due to the hydrostatic pressure of the water.



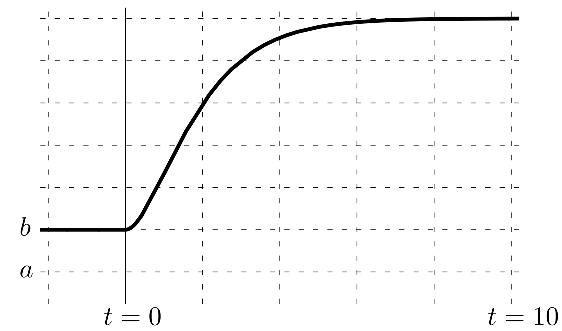
4. [12 points] Zeno's paradox says that one can never arrive somewhere because one must always first travel half-way there—and that having traveled half-way there, one must travel half of the remaining distance, then half of the distance remaining after that, etc.

(a) [4 points of 12] Suppose that you start with the goal of traveling 20 km. Let d_n be the total distance that you have gone after having traveled the nth half-distance to your goal. Find d_1, d_2, d_3 and d_4.

(b) [6 points of 12] Find a closed-form expression for the distance you've traveled after n half-distances.

(c) [2 points of 12] What is the sum as the number of half-distances traveled goes to infinity? (That is, how far do you travel if you continue "forever"?)

5. [16 points] For all parts of this problem, refer to the graph to the right, which gives a cumulative distribution function $P(t)$ for some density function $p(x)$. The given graph shows all important features of the distribution (for values of t greater and less than those shown, the behavior shown continues).



(a) [4 points of 16] What are the y-values a and b? Why?

(b) [4 points of 16] What is the approximate value of the median of this distribution?

(c) [4 points of 16] Suppose that two points on the graph are $(3.9, 0.90)$ and $(4.1, 0.92)$. Estimate $p(4)$.

(d) [4 points of 16] Continue to suppose that two points on the graph are $(3.9, 0.90)$ and $(4.1, 0.92)$. Estimate $\int_0^4 p(x)\,dx$.

6. [12 points] Recall that the Great Pyramid of Giza was (originally) approximately 480 ft high, with a square base approximately 760 ft to a side. The pyramid was made of close to 2.4 million limestone blocks, and has several chambers and halls that extended into its center. It is not too far from the truth to suppose that these open areas are located along the vertical centerline of the pyramid, and that we can therefore think of the density of the pyramid varying only along its vertical dimension. Suppose that the result is that the density of the pyramid is approximately $\delta(h) = (0.00011(h - 240)^2 + 134.2)$ lb/ft^3, where h is the height measured up from the base of the pyramid.

(a) [6 points of 12] Set up an integral to find the weight W of the pyramid. You need not evaluate the integral to find the actual weight.

(b) [6 points of 12] Give an expression, in terms of integral(s), that tells how far off the ground the center of mass of the pyramid is. Again, you need not evaluate the integral(s). *(Note that you may set up the expression in terms of the weight density without worrying about converting it to a mass density.)*

7. [9 points] Suppose that you invest \$5000 in a savings account that pays 2.5% interest, compounded annually. At the end of each year you withdraw the interest made on the principal in the account, and then reinvest \$100. Find a formula for R_n, the return (the amount that you take take home, after the reinvestment) from the account at the end of nth year.

8. [8 points] Consider the area whose boundary is given in polar coordinates by the equations $\theta = \frac{\pi}{3}$ and $r = f(\theta)$. The continuous function $f(\theta)$ is defined for $\pi/3 \le \theta \le 3\pi/2$, and values of this function (spaced $\Delta\theta = 7\pi/24$ apart) are given in the table below.

$\theta =$	$\pi/3$	$5\pi/8$	$11\pi/12$	$29\pi/24$	$3\pi/2$
$f(\theta) =$	1.866	1.924	1.249	0.3912	0

Give a reasonably accurate estimate of the area of this region.

9. [15 points] Suppose that we know that $\sum_{n=1}^{\infty} a_n$ converges—but we don't know what a_n is. For each of the series below, determine whether it converges, diverges, or we cannot tell (that is, there could be one value for a_n that would converge and one that would not). Circle your answer and provide a short but careful explanation for your answer (how do we know the series converges or diverges?, or, what examples show that we cannot tell?).

(a) [3 points of 15] $\sum_{n=1}^{\infty} |a_n|$ converges diverges cannot tell

(b) [3 points of 15] $\sum_{n=1}^{\infty} (-1)^n |a_n|$ converges diverges cannot tell

(c) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n+1}{a_n+5}$ converges diverges cannot tell

(d) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges diverges cannot tell

(e) [3 points of 15] $\sum_{n=1}^{\infty} \frac{3^n a_n}{n^3}$ converges diverges cannot tell